CARTAN ON CARTAN-KÄHLER

LECTURE 16, MATH 8234

This note is based on sections **63-66** *in É. Cartan's EDS book.*

Cartan's Statement:

Assumptions:

$(M^n, \mathcal{I})$: real analytic EDS

M: connected

$\mathcal{I}$: containing no degree-0 part

Suppose that $r(E_{p-1}) = 0$ for all regular E_{p-1}

$(E_p)_O$: an ordinary integral element based at $O \in M$

$(E_{p-1})_O$ regular, contained in $(E_p)_O$

V_{p-1} real analytic (regular) integral manifold tangent to $(E_{p-1})_O$

Conclusion:

There exists a unique real analytic integral manifold $V_p \supset V_{p-1}$ tangent to $(E_p)_O$.

In his EDS book, Cartan proved the case when $p = 3$.

Setting:

Can choose local coordinates (x^i, z^λ) $(i = 1, 2, 3; \lambda = 1, 2, \ldots, n-3)$ such that:

O: $x^i = 0$, $z^\lambda = a^\lambda$

$(E_3)_O$: $\mathrm{d}z^\lambda = a_i^\lambda \mathrm{d}x^i$

$(E_2)_O$: $\mathrm{d}z^\lambda = a_i^\lambda \mathrm{d}x^i$, $\mathrm{d}x^3 = 0$

V_2: $x^3 = 0$, $z^\lambda = \Phi^\lambda(x^1, x^2)$ subject to

$$\Phi^\lambda(0,0) = a^\lambda, \quad \partial_1 \Phi^\lambda|_{(0,0)} = a_1^\lambda, \quad \partial_2 \Phi^\lambda|_{(0,0)} = a_2^\lambda$$

Goal:

Look for a 3-dimensional integral manifold V_3 given by

$$z^\lambda = F^\lambda(x^1, x^2, x^3)$$

with

$$F^\lambda(x^1, x^2, 0) = \Phi^\lambda(x^1, x^2) \qquad \text{(i.e., } V_2 \subset V_3\text{)}$$

and

$$\partial_3 F^\lambda(0,0,0) = a_3^\lambda \qquad \text{(i.e., } V_3 \text{ is tangent to } (E_3)_O\text{)}$$

Further Set-up:

In coordinates, suppose that $\mathcal{I}$ is algebraically generated by:

1-forms:

$$\theta_\alpha = A_{\alpha i} \mathrm{d}x^i + A_{\alpha\lambda} \mathrm{d}z^\lambda$$

2-forms:

$$\phi_\alpha = \frac{1}{2} A_{\alpha ij} \mathrm{d}x^i \wedge \mathrm{d}x^j + A_{\alpha i\lambda} \mathrm{d}x^i \wedge \mathrm{d}z^\lambda + + \frac{1}{2} A_{\alpha\lambda\mu} \mathrm{d}z^\lambda \wedge \mathrm{d}z^\mu$$

$$(A_{\alpha ij} = -A_{\alpha ji}, \quad A_{\alpha\lambda\mu} = -A_{\alpha\mu\lambda})$$

assuming that $\mathrm{d}\theta_\alpha$ occur in ϕ_α

3-forms:

$$\psi_\alpha = \frac{1}{6}A_{\alpha ijk}\mathrm{d}x^{ijk} + \frac{1}{2}A_{\alpha ij\lambda}\mathrm{d}x^{ij} \wedge \mathrm{d}z^\lambda + \frac{1}{2}A_{\alpha i\lambda\mu}\mathrm{d}x^i \wedge \mathrm{d}z^{\lambda\mu} + \frac{1}{6}A_{\alpha\lambda\mu\nu}\mathrm{d}z^{\lambda\mu\nu}$$
$$(A_{\alpha ijk} = -A_{\alpha jik} = -A_{\alpha ikj}, \text{ etc.})$$

assuming that $\mathrm{d}\phi_\alpha$ occur in ψ_α

Therefore, V_3 is an integral manifold if and only if the following expressions vanish:

$$H_{\alpha i} := A_{\alpha i} + A_{\alpha\lambda}\frac{\partial z^\lambda}{\partial x^i}$$
$$H_{\alpha ij} := A_{\alpha ij} + A_{\alpha i\lambda}\frac{\partial z^\lambda}{\partial x^j} - A_{\alpha j\lambda}\frac{\partial z^\lambda}{\partial x^i} + A_{\alpha\lambda\mu}\frac{\partial z^\lambda}{\partial x^i}\frac{\partial z^\mu}{\partial x^j}$$
$$H_{\alpha 123} := A_{\alpha 123} + \sum_{\substack{(ijk) \text{ even} \\ \text{perm. of } (123)}} A_{\alpha ij\lambda}\frac{\partial z^\lambda}{\partial x^k} + \sum_{\substack{(ijk) \text{ even} \\ \text{perm. of } (123)}} A_{\alpha i\lambda\mu}\frac{\partial z^\lambda}{\partial x^j}\frac{\partial z^\mu}{\partial x^k} + A_{\alpha\lambda\mu\nu}\frac{\partial z^\lambda}{\partial x^1}\frac{\partial z^\mu}{\partial x^2}\frac{\partial z^\nu}{\partial x^3}$$

Argument:

Group the H's into two classes:

$$(A) : H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}$$
$$(B) : H_{\alpha 3}, H_{\alpha 13}, H_{\alpha 23}, H_{\alpha 123}$$

Simple fact from linear algebra: If a matrix A (depending on parameters p^α) has rank r at (p_0^α), then, in a neighborhood of (p_0^α), $\mathrm{rank}(A) \geqslant r$.

Simple fact from calculus: If, on $\mathbb{R}^n$, whenever $x^1 = x^2 = 0$, f is a linear function in x^3 without a constant term, then f is of the form

$$f = Ax^1 + Bx^2 + Cx^3$$

where C is a function of x^1, x^2, and A, B are functions on $\mathbb{R}^n$.

When

$$x^i = 0, \quad z^\lambda = a^\lambda, \quad \frac{\partial z^\lambda}{\partial x^1} = a_1^\lambda, \quad \frac{\partial z^\lambda}{\partial x^2} = a_2^\lambda, \tag{1}$$

the expressions in (A) are identically zero. (This is because $(E_2)_O$ is an integral element.)

Thus the polar equations are given by substituting (1) in the expressions in (B), which become linear in $\partial_3 z^\lambda$. The rank of these expressions must be $n-3$ since we have assumed $r(E_2) = 0$. So we can pick a basis, the corresponding H's being denoted

$$H_{\rho 3}, H_{\rho 13}, H_{\rho 23}, H_{\rho 123}.$$

Cartan called these expressions the *principals.* The rest in (B) are called *non-principals.* He denoted the non-principals by

$$H_{\alpha' 3}, H_{\alpha' 13}, H_{\alpha' 23}, H_{\alpha' 123}.$$

For values of $x^i, z^\lambda, \partial_1 z^\lambda, \partial_2 z^\lambda$ near (1), whenever they (together with the condition $\mathrm{d}x^3 = 0$) determine a 2-dimensional integral element, the integral element will be regular, and the rank of the polar equations (i.e., equations obtained by substituting those values in (B)) would be $n-3$, by regularity; they are spanned by the 'principals' (substitution made), by the linear algebra fact mentioned above.

Since there are $n-3$ principals, it is not hard to see that the non-principals are linear combinations of

$$H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}, H_{\rho 3}, H_{\rho 13}, H_{\rho 23}, H_{\rho 123}$$

with coefficients being functions of $x^i, z^\lambda, \partial_1 z^\lambda, \partial_2 z^\lambda$. (**Exercise:** Justify this.)
In particular, each $H_{\alpha' 3}$ can only be a linear combination of $H_{\rho 3}$, because it has degree-1 in $\partial_i z^\lambda$ and is independent of $\partial_1 z^\lambda$ and $\partial_2 z^\lambda$.

First use of Cauchy-Kowalewski:
The vanishing of the $n-3$ principals in the $n-3$ z^λ's give rise to a determined system of Cauchy form:

$$\text{PDE}: \qquad \frac{\partial z^\lambda}{\partial x^3} = (\text{analytic expression in } x^i, z^\lambda, \partial_1 z^\lambda, \partial_2 z^\lambda),$$
$$\text{I.C.} \qquad z^\lambda(x^1, x^2, 0) = \Phi^\lambda(x^1, x^2).$$

By Cauchy-Kowalewski, we obtain a unique solution $z^\lambda(x^1, x^2, x^3)$. Let's call the corresponding integral manifold V_3.

Is V_3 an integral manifold of $(M, \mathcal{I})$?
It suffices to verify that all the H's vanish on V_3. Because the principals vanish identically on V_3, it suffices to verify that

$$H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}$$

vanish on V_3.

Using $\mathcal{I}$ being differentially closed:
On V_3, $H_{\alpha 3}$ are identically zero, since they are linear combinations of the principals $H_{\rho 3}$. Therefore, on V_3,

$$\theta_\alpha = H_{\alpha 1} \mathrm{d}x^1 + H_{\alpha 2} \mathrm{d}x^2.$$

Differentiating this, we obtain

$$\mathrm{d}\theta_\alpha = -\partial_3 H_{\alpha 1} \mathrm{d}x^1 \wedge \mathrm{d}x^3 - \partial_3 H_{\alpha 2} \mathrm{d}x^2 \wedge \mathrm{d}x^3 + (\partial_1 H_{\alpha 2} - \partial_2 H_{\alpha 1}) \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

Since $\mathrm{d}\theta_\alpha$ occur in ϕ_α, $\partial_3 H_{\alpha 1}$ occurs in $H_{\alpha 13}$ and is therefore a linear combination of $H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}$.
Similarly, $\partial_3 H_{\alpha 2}$ occurs in $H_{\alpha 23}$ and is also a linear combination of $H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}$.
Now, on V_3,

$$\phi_\alpha = H_{\alpha 12} \mathrm{d}x^1 \wedge \mathrm{d}x^2 + H_{\alpha 23} \mathrm{d}x^2 \wedge \mathrm{d}x^3 + H_{\alpha 13} \mathrm{d}x^1 \wedge \mathrm{d}x^3.$$
$$\mathrm{d}\phi_\alpha = (\partial_3 H_{\alpha 12} + \partial_1 H_{\alpha 23} - \partial_2 H_{\alpha 13}) \mathrm{d}x^{123}.$$

Therefore, $\partial_3 H_{\alpha 12} + \partial_1 H_{\alpha 23} - \partial_2 H_{\alpha 13}$ occurs in $H_{\alpha 123}$ and is, therefore, a linear combination of $H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}$.
Since, on V_3, $H_{\alpha 23}$ and $H_{\alpha 13}$ are linear combinations of $H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}$ with coefficients being analytic functions in x^i, $\partial_3 H_{\alpha 12}$ is a linear combination of

$$H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}, \partial_1 H_{\alpha 1}, \partial_1 H_{\alpha 2}, \partial_1 H_{\alpha 12}, \partial_2 H_{\alpha 1}, \partial_2 H_{\alpha 2}, \partial_2 H_{\alpha 12}$$

Second use of Cauchy-Kowalewski:
In sum, on V_3, $\partial_3 H_{\alpha 1}, \partial_3 H_{\alpha 2}, \partial_3 H_{\alpha 12}$ are all linear combinations of

$$H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}, \partial_1 H_{\alpha 1}, \partial_1 H_{\alpha 2}, \partial_1 H_{\alpha 12}, \partial_2 H_{\alpha 1}, \partial_2 H_{\alpha 2}, \partial_2 H_{\alpha 12}$$

with coefficients being analytic functions in x^1, x^2, x^3, the initial conditions being

$$H_{\alpha 1}(x^1, x^2, 0) = H_{\alpha 2}(x^1, x^2, 0) = H_{\alpha 12}(x^1, x^2, 0) = 0.$$

Thus, $H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}$ satisfy a determined Cauchy problem. By the uniqueness part of Cauchy-Kowalewski, $H_{\alpha 1}, H_{\alpha 2}, H_{\alpha 12}$ are identically zero on V_3.

This completes the proof.