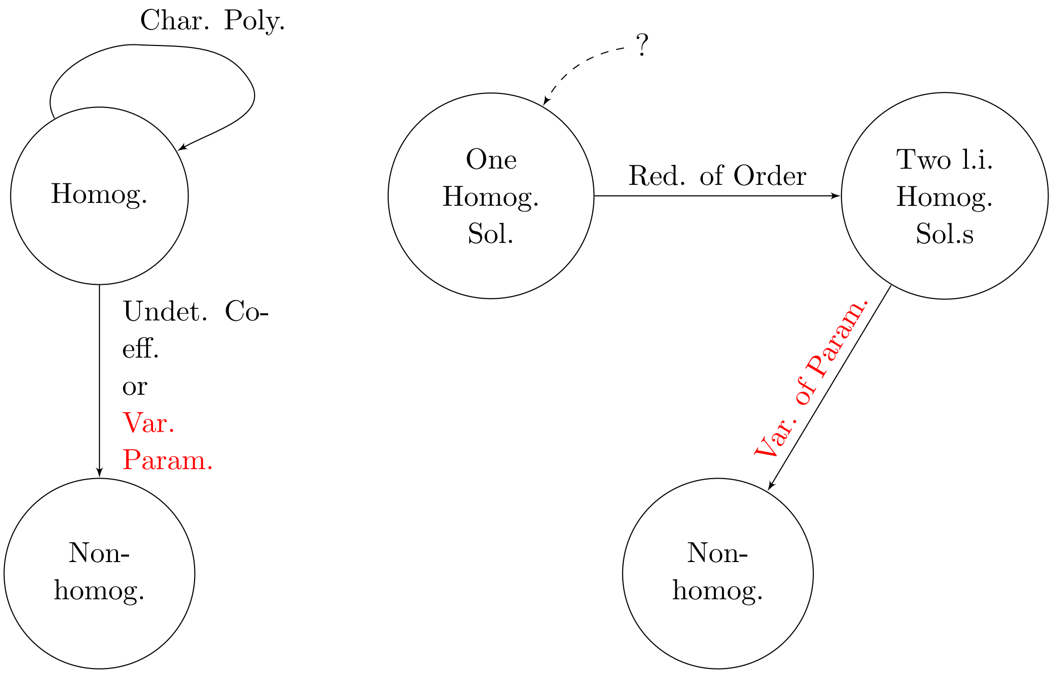
LECTURE 9: VARIATION OF PARAMETERS

1. Picturing Second Order Linear ODEs

By the previous discussion, the second order linear ODEs can be organized in the diagram below, which shows various techniques that can be applied for finding solutions. For example, the constant coefficient homogeneous equations are solved using *characteristic polynomials*. With a nonzero solutions known, a homogeneous equation can be completely solved by the method of *reduction of order*. For certain types of non-homogeneous constant-coefficient equations, a particular solution can be found by "guessing" the form of the solution, known as the method of *undetermined coefficients* . Yet there are still some cases which we haven't dealt with.

- For a constant coefficient non-homogeneous equation of the form
$$y'' + ay' + b = g(x),$$
where the function $g(x)$ is not a linear combination or product of polynomials, exponential functions and sine/cosine functions, one may not be able to "guess" the form of a particular solution.
- How to find a particular solution for a non-constant coefficient, non-homogeneous equation?
- Find a nonzero solution for a non-constant coefficient homogeneous equation.



(A) Constant-coefficient (B) Nonconstant-coefficient

Figure 1. Second Order Linear ODEs

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For the last problem listed above, unfortunately, there is no general theory to resolve it. However, for the first two problems, the method of *variation of parameters* turns out to be very helpful.

2. Variation of Parameters

The idea behind *variation of parameters* is, in a way, similar to that behind the method of *reduction of order*. The latter asks whether it is possible to multiply a known homogeneous solution by some function to obtain another (independent) homogeneous solution. The former (variation of parameters) supposes that a fundamental set of solutions is known for the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

say, $\alpha(x), \beta(x)$, then asks, is it possible to find functions $u(x), v(x)$ such that

$$u(x)\alpha(x) + v(x)\beta(x)$$

is a solution of the non-homogeneous equation

$$y'' + p(x)y' + q(x)y = r(x)?$$

The testing of this idea is of course plugging the expression $y(x) = u(x)\alpha(x) + v(x)\beta(x)$ in the non-homogeneous equation above, and see what condition is imposed on the functions $u(x)$ and $v(x)$.

In the following calculation, I'll drop all the independent variable argument in functions to make thing look cleaner. For instance, I'd write $x^2 f$ instead of $x^2 f(x)$. Since x is the only independent variable, this causes no confusion.

We need

$$\begin{aligned} r &= (u\alpha + v\beta)'' + p(u\alpha + v\beta)' + q(u\alpha + v\beta) \\ &= (u''\alpha + 2u'\alpha' + u\alpha'' + v''\beta + 2v'\beta' + v\beta'') \\ &\qquad + (pu'\alpha + pu\alpha' + pv'\beta + pv\beta') \\ &\qquad\qquad + (qu\alpha + qv\beta) \\ &= (u''\alpha + v''\beta) + 2(u'\alpha' + v'\beta') + p(u'\alpha + v'\beta). \end{aligned}$$

In particular, note that the terms in red sum up to vanish identically, since α satisfies

$$\alpha'' + p\alpha' + q = 0.$$

Similarly, the terms in blue sum up to vanish.

Now we have a condition on u, v, namely,

$$\text{(C)} \qquad (u''\alpha + v''\beta) + 2(u'\alpha' + v'\beta') + p(u'\alpha + v'\beta) = r.$$

This is in contrast to how the method of reduction of order works. In reduction of order, calculation leads to a separable equation for the derivative of the unknown multiplier. Here, however, we have two functions u, v to determine, which are constrained by a *single differential equation*.[1] One may feel that a "single" equation may be too few, comparing to the number of unknowns: two; and that a "differential" equation may be something less desirable than an algebraic equation. This motivates the idea

- Impose another relation between u, v;

[1]Even if we reduce the order in (C), noting that it only involves the derivatives of u, v, we'd still get a differential equation in u', v'

- If possible, choose the imposed relation properly so that relation (C) is reduced to an algebraic relation between u', v'.

Since

$$(u'\alpha + v'\beta)' = (u''\alpha + v''\beta) + (u'\alpha' + v'\beta'),$$

we can replace the term $(u''\alpha + v''\beta)$ in condition (C) by $(u'\alpha + v'\beta)' - (u'\alpha' + v'\beta')$, obtaining

$$(u'\alpha + v'\beta)' - (u'\alpha' + v'\beta') + 2(u'\alpha' + v'\beta') + p(u'\alpha + v'\beta) = r,$$

or

$$(u'\alpha + v'\beta)' + (u'\alpha' + v'\beta') + p(u'\alpha + v'\beta) = r$$

Now it is clear that if we impose

$$u'\alpha + v'\beta = 0,$$

condition (C) then reduces to

$$u'\alpha' + v'\beta' = r.$$

Hey, these two equations are simply (algebraic) linear equations in u', v', so we can write them together in a matrix form:

$$\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}.$$

Note that the coefficient matrix is invertible everywhere, whose determinant is simply the Wronskian $W(x)$, hence u', v' can be solved for by linear algebra:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W} \begin{pmatrix} \beta' & -\beta \\ -\alpha' & \alpha \end{pmatrix} \begin{pmatrix} 0 \\ r \end{pmatrix}.$$

To sum up (and putting back the independent variable x), a particular solution of the non-homogeneous equation

$$y'' + p(x)y' + q(x)y = r(x)$$

is given by

$$y_p(x) = \left(-\int \frac{\beta(x)r(x)}{W(\alpha,\beta)} dx \right) \alpha(x) + \left(\int \frac{\alpha(x)r(x)}{W(\alpha,\beta)} dx \right) \beta(x).$$

3. Applications

Example 1. The constant coefficient non-homogeneous equation $y'' + y = \frac{1}{\sin x}$ does not fit in the usual method of undetermined coefficients. However, general solutions may be found using variation of parameters. Note that $\alpha(x) = \cos x$, $\beta(x) = \sin x$ are linearly independent homogeneous solutions. And we calculate that

$$W(\alpha, \beta) = \det \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} = 1.$$

Hence, by the method of variation of parameters, the non-homogeneous equation above has a particular solution

$$\begin{aligned} y_p(x) &= \left(-\int \sin x \frac{1}{\sin x} dx\right) \cos x + \left(\int \frac{\cos x}{\sin x} dx\right) \sin x \\ &= -x\cos x + \sin x \ln(\sin x). \end{aligned}$$

Example 2. Consider the non-constant coefficient, non-homogeneous equation

$$(x^2-1)y'' - 2xy' + 2y = (x^2-1)^2.$$

The corresponding homogeneous equation

$$(x^2-1)y'' - 2xy' + 2y = 0$$

has the obvious solution

$$\alpha(x) = x.$$

By the method of reduction of order, suppose another solution to be in the form $xw(x)$, then after a brief calculation, we see that $w(x)$ must satisfy

$$(x^3-x)w'' - 2w' = 0.$$

Hence,

$$\frac{w''}{w'} = \frac{2}{x^3-x} = \frac{-2}{x} + \frac{1}{x+1} + \frac{1}{x-1}.$$

This implies that

$$w' = \frac{(x+1)(x-1)}{x^2} = 1 - \frac{1}{x^2},$$

and the choice

$$w = x + \frac{1}{x}.$$

Therefore,

$$\beta(x) = x \cdot w(x) = x^2 + 1$$

is another homogeneous solution.

Note that the Wronskian

$$W(\alpha, \beta) = \det \begin{pmatrix} x & x^2+1 \\ 1 & 2x \end{pmatrix} = x^2 - 1.$$

Finally, for a particular solution of the non-homogeneous equation, we apply the method of variation of parameters, noting that $r(x) = x^2 - 1$, and obtaining

$$\begin{aligned} y_p(x) &= \left(-\int \frac{(x^2+1)(x^2-1)}{x^2-1} dx\right) x + \left(\int \frac{x(x^2-1)}{x^2-1} dx\right) (x^2+1) \\ &= \frac{1}{6}x^4 - \frac{1}{2}x^2. \end{aligned}$$

The general solutions are then

$$y(x) = \frac{1}{6}x^4 - \frac{1}{2}x^2 + c_1 x + c_2(x^2+1).$$