LECTURE 8: SECOND ORDER LINEAR ODES

1. General Theory

It is clear that, generically, a second order linear ODE takes the form:

$$y'' + p(x)y' + q(x)y = g(x).$$

This equation is said to be *homogeneous* if $g(x) \equiv 0$.

Of course, the theoretical foundation of solving ODEs of this type is the Existence and Uniqueness Theorem.

Theorem (Existence & Uniqueness) For the initial value problem

$$y'' + p(x)y' + q(x)y = g(x), \qquad y(t_0) = y_0, y'(t_0) = d_0,$$

if $p(x), q(x), g(x)$ are continuous within some interval I containing t_0, then solution exists and is unique on the interval I.

Linearity of the equation implies that one has the *principle of superposition* of solutions. That is, if there are functions $y_1(x), y_2(x)$ satisfying

$$y_1'' + p(x)y_1' + q(x)y_1 = g_1(x),$$

$$y_2'' + p(x)y_2' + q(x)y_2 = g_2(x),$$

then $\lambda y_1(x) + \mu y_2(x)$ is a solution to the equation

$$y'' + p(x)y' + q(x)y = g_1(x) + g_2(x).$$

For this very reason, all solutions of a homogeneous linear second order ODE form a vector space, a basis of which is called *fundamental solutions*. Similarly, all the solutions of a non-homogeneous equation is the superposition of a particular solution with the solutions of the corresponding homogeneous equation (i.e., all the solutions of a non-homogeneous equation form an *affine space*).

Given a homogeneous second order linear ODE which satisfies the hypotheses in the above theorem, the conclusion of the theorem together with the principle of superposition implies that the vector space of solutions is exactly 2-dimensional. Thus, one only needs to find two non-zero solutions which are linearly independent. As we learned before, a test for linear independence between solutions y_1, y_2 can be carried out by the *Wronskian*:

$$W(x) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix},$$

and the Wronskian is either never zero or constantly zero (review linear algebra).

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2. Recipes for Solutions

When it comes to practically solving a second order linear ODE, it turns out useful to ask whether the equation is homogeneous/non-homogeneous, constant-coefficient/non-constant-coefficient.

2.1. Constant-coefficient, Homogeneous.

We are familiar with solving homogeneous, constant-coefficient equations using *characteristic polynomials.* For example, the equation

$$y'' + 3y' + \tfrac{25}{4}y = 0$$

has the characteristic polynomial

$$p(\lambda) = \lambda^2 + 3\lambda + \tfrac{25}{4},$$

which has a pair of complex conjugate roots

$$\lambda_1 = \frac{-3+4i}{2}, \quad \lambda_2 = \frac{-3-4i}{2}.$$

Hence a complex solution can be found:

$$z(x) = e^{(-3+4i)x/2}.$$

Furthermore, one could verify that if $z(x)$ is a complex solution, the its real and imaginary parts are both solutions of the original equation. Thus the real fundamental solutions are

$$y_1(x) = e^{-3x/2}\cos 2x, \quad y_2(x) = e^{-3x/2}\sin 2x.$$

Generally, if the characteristic polynomial has repeated (real) roots λ, we have fundamental solutions

$$y_1(x) = e^{\lambda x}, \quad y_2(x) = xe^{\lambda x}.$$

2.2. Constant-coefficient, Non-homogeneous.

To be precise, the non-homogeneity refers to the term $g(x)$ being an exponential, a sine or cosine, a polynomial or certain combination of such functions. To find a particular solution, one need to make a guess of the form of solution, with coefficients undetermined; then plug the guessed "solution" to the equation to establish relations for solving the those coefficients. Indeed, we have the

Theorem. Consider the constant coefficient linear equation

$$ay'' + by' + cy = g(x),$$

with the characteristic polynomial $p(\lambda) = a\lambda^2 + b\lambda + c$.

(1) If $g(x) = P_n(x)$ is a polynomial of degree n, then $y_p = x^s(a_nx^n + ... + a_1x + a_0)$, where s is the multiplicity of 0 as a root of $p(\lambda)$.

(2) If $g(x) = P_n(x)e^{rx}$, then $y_p = x^s(a_nx^n + ... + a_1x + a_0)e^{rt}$, where s is the multiplicity of r as a root of $p(\lambda)$, n is the degree of $P_n(x)$.

(3) If $g(x) = P_n(x)e^{ax}\cos bx$ or $g(x) = P_n(x)e^{ax}\sin bx$, then $y_p = x^s[(a_nx^n + ... + a_1x + a_0)e^{at}\cos bx + (b_nx^n + ... + b_1x + b_0)e^{at}\sin bx]$, where s is the multiplicity of $a + ib$ as a root of $p(\lambda)$.

For example, the right hand side of the equation

$$y'' + 10y' + 25y = 12e^{-5x}$$

falls into category (2) of the theorem, where $r = -5$ is a root of multiplicity 2 of the characteristic polynomial. The polynomial $P_n(x) = 1$ has degree zero. Hence we make the guess

$$y(x) = Ax^2e^{-5x}.$$

Plugging in the equation and simplifying, we obtain $A = 6$, and hence

$$y(x) = 6x^2e^{-5x}$$

is a particular solution.

2.3. **Non-constant Coefficient, Homogeneous.**

In general, there is no easy way of finding a solution. However, if a solution is spotted, then a fundamental set of solutions can be found by the method of *reduction of order*, which will be discussed in the following section.

2.4. **Non-constant Coefficient, Non-homogeneous.**

This seems the hardest case. The good news is, a particular solution can be found as long as a fundamental set of solutions of the corresponding homogeneous equation is given. The technique is called *variation of parameters*, which is the topic of the next lecture.

3. Repeated Roots, Reduction of Order

The idea of *reduction of order* is very simple: Given $y_1(x)$, a known solution to a homogeneous equation, is it possible to obtain another solution in the form $v(x)y_1(x)$, where v is some non-constant function?

For the homogeneous equation

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0,$$

$y(x) = x$ is easily seen to be a solution. To test the condition for $v(x)y(x)$ to be a solution, replace y with vy in the equation, obtaining:

$$\begin{aligned}
0 &= (vy)'' + \frac{1}{x}(vy)' - \frac{1}{x^2}(vy) \\
&= v''y + 2v'y' + vy'' + \frac{1}{x}v'y + \frac{1}{x}vy' - \frac{1}{x^2}vy \\
&= v''y + 2v'y' + \frac{1}{x}v'y + \left(vy'' + \frac{1}{x}vy' - \frac{1}{x^2}vy\right) \\
&= v''x + 3v'.
\end{aligned}$$

Note that the condition for v involves only its derivatives, hence we could make the substitution $z = v'$, and the condition for v translates to one in z:

$$z'x + 3z = 0.$$

By separation of variables,

$$z = x^{-3},$$

hence

$$v = \int z(x)dx = \frac{-1}{2}x^{-2},$$

and another solution of the original equation is

$$vy = -\frac{1}{2x},$$

which is clearly independent from $y(x) = x$.

In general, suppose that $y(x)$ is known for solving the equation $y'' + p(x)y' + q(x)y = 0$. Let $v(x)y(x)$ be another solution. Then we have:

$$\begin{aligned}
0 =& (vy)'' + p(vy)' + q(vy) \\
=& (v'y + vy')' + p(v'y + vy') + q(vy) \\
=& (v''y + 2v'y' + vy'') + p(v'y + vy') + q(vy) \\
=& v''y + (2y' + py)v' + v(y'' + py' + qy) \\
=& v''y + (2y' + py)v'.
\end{aligned}$$

Thus, $z = v'(x)$ satisfies the first order separable equation:

$$z'y + (2y' + py)z = 0.$$

Thus, z can be solved. Finally, $v(x)$ can be found by a direct integration of $z(x)$.

Exercise. The method of reduction of order also applies to the constant coefficient (homogeneous) case, when the characteristic polynomial has repeated roots λ. One solution is obvious: $e^{\lambda x}$. Find the fundamental set of solutions.