

LECTURE 7: EULER'S METHOD

In the experiment below, we are going to explore a numerical method for solving first order ODEs, called Euler's Method. The idea is simple: use continuous line segments to approximate solution curves. To be more precise, supposing that (x_0, y_0) is on the solution curve, then for small Δx , we could use $y_0 + \Delta x \cdot f(x_0, y_0)$ as an approximation of $y(x_0 + \Delta t)$. Euler's method suggests us continuing this process, taking $(x_1, y_1) = (x_0 + \Delta x, y_0 + \Delta x \cdot f(x_0, y_0))$ as the new starting point, using the accurate tangent slope at this point, and carry out the approximation to obtain $(x_2, y_2) = (x_1 + \Delta x, y_1 + \Delta x \cdot f(x_1, y_1))$, and so on. A pseudocode for this method is as follows.

```
procedure EULER'S METHOD FOR EQUATION  $\frac{dy}{dx} = f(x, y)$ 
   $(x_0, y_0) \leftarrow initialValue$ 
   $\Delta x \leftarrow stepSize$ 
   $N \leftarrow numSteps$ 
  loop:
    for  $(i = 0, i < N, i++)$  do
       $x_{i+1} = x_i + \Delta x;$ 
       $y_{i+1} = y_i + f(x_i, y_i)\Delta x;$ 
```

For the initial value problem

$$\frac{dy}{dx} = 1 - x - 0.5y, \quad y(0) = 1,$$

we use Mathematica[®] to carry out the Euler's method for step sizes $\Delta x = 1, 0.5, 0.1$ to approximate the solution within $\{0 \leq x \leq 5\}$. The code and the result are shown below, where the orange and blue curves represent accurate solutions and approximate solutions, respectively.

```
procedure MATHEMATICA® FOR  $\frac{dy}{dx} = 1 - x - 0.5y, y(0) = 1$ 
  Clear[y];
  y[0]=1;
  stepSize=1; # set to be 0.5, 0.1 later on
  Do[y[n+1]=y[n]+stepSize*(1-stepSize*n-0.5*y[n]),{n,0,5/stepSize-1}];
  y[5/stepSize];

  table=Table[{N[stepSize*k],y[k]},{k,0,5/stepSize}]
  a=ListPlot[table, Joined→True]; #Plotting the piecewise-linear approximation
  b=VectorPlot[{1,1-x-0.5*y},{x,0,5},{y,-5,5}]; #Plotting the vector field
  c=Plot[{-2e-0.5x(2.5-3e0.5x+e0.5xx)},{x,0,5}, PlotStyle→Orange]; #Plotting the
  accurate solution

  Show[a,b,c]
```

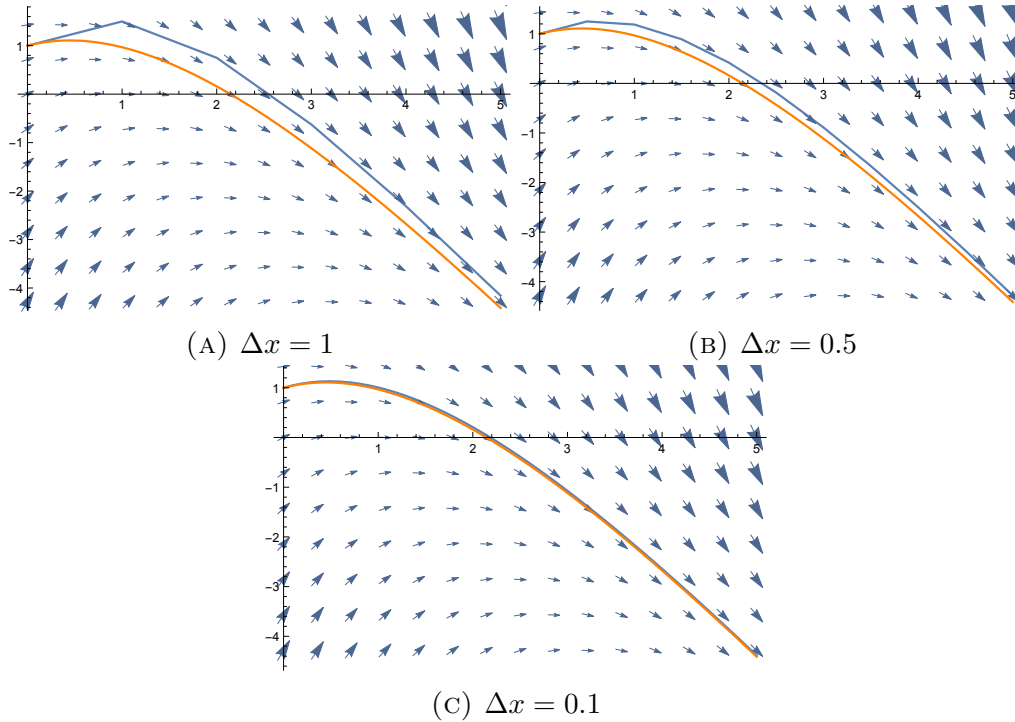


FIGURE 1. $\frac{dy}{dx} = 1 - x - 0.5y$, $y(0) = 1$

Next, for the initial value problem slightly modified from the one above (with only a sign change),

$$\frac{dy}{dx} = 1 - x + 0.5y, \quad y(0) = 1,$$

we have

```

procedure MATHEMATICA® FOR  $\frac{dy}{dx} = 1 - x + 0.5y, y(0) = 1$ 
  Clear[y];
  y[0]=1;
  stepSize=1; # set to be 0.5, 0.1,0.025 later on
  Do[y[n+1]=y[n]+stepSize*(1-stepSize*n+0.5*y[n]),{n,0,5/stepSize-1}];
  y[5/stepSize];

  table=Table[{N[stepSize*k],y[k]},{k,0,5/stepSize}]
  a=ListPlot[table, Joined→True]; #Plotting the piecewise-linear approximation
  b=VectorPlot[{1,1-x+0.5*y},{x,0,5},{y,-5,5}]; #Plotting the vector field
  c=Plot[{2 - e0.5x + 2x},{x,0,5}, PlotStyle→Orange]; #Plotting the accurate solu-
tion

  Show[a,b,c]

```

By looking at the figures, you may have, but are not restricted to, the following four observations:

(1) The piecewise-linear approximations have better accuracy when we choose smaller step sizes.

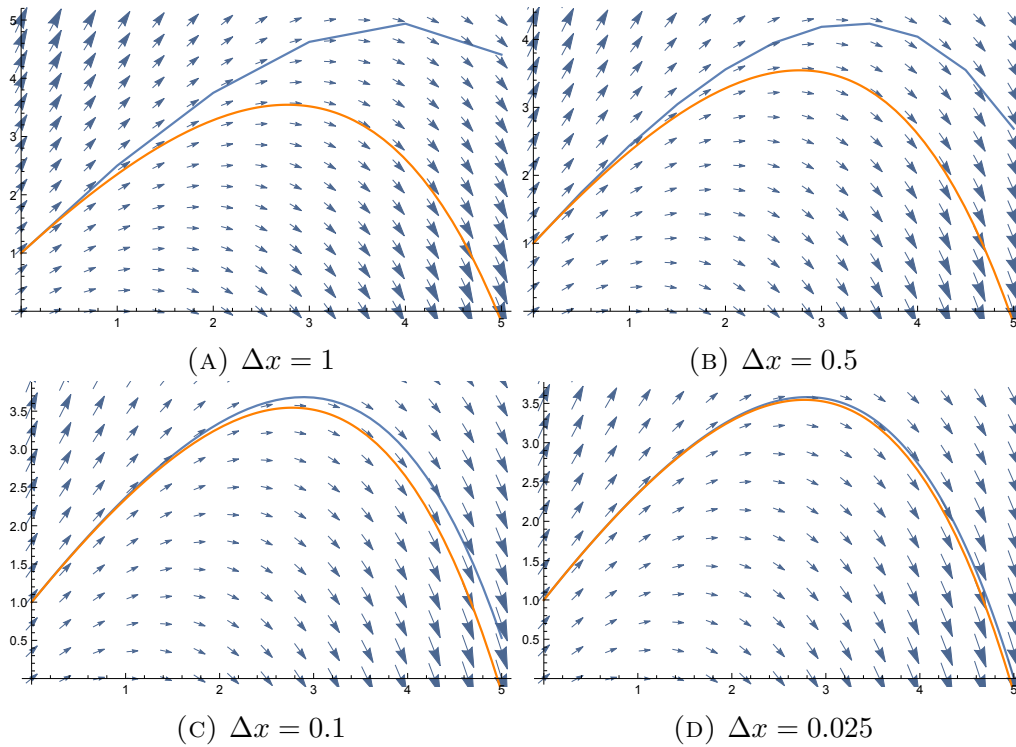


FIGURE 2. $\frac{dy}{dx} = 1 - x + 0.5y$, $y(0) = 1$

(2) Within each line segment of the piecewise-linear approximation, the slope at the starting point is always accurate, even though the starting point itself may not be accurate.

(3) Euler's method works better for the first equation than the second.

(4) The vector field for the first equation appears “converging” while that for the latter appears “diverging”.

For the first observation, you may want a rigorous proof. Here, I recommend you solve through Ex.20, Sec. 2.7. For the second observation, we have more to say. The accuracy of the slope is just how we define the Euler's method. The inaccuracy of the end points are exactly the reason why error might accumulate during the procedure. Combining with the fourth observation, we see that when the vector field diverges, one has to pay more for the inaccuracy occurred the middle, and the error at the end could be large, unless we choose Δx to be really, really small (Figure 2).