# LECTURE 6: EXACT EQUATIONS AND INTEGRATING FACTORS

## 1. Exact Equations

The first order ODEs, which we usually write as

$$\frac{dy}{dx} = f(x, y)$$

can also be put in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0.$$

This new form of equation is said to be **exact** if there exists a function $F(x, y)$, such that

$$F_x(x, y) = M(x, y), \quad F_y(x, y) = N(x, y).$$

**Motivation**: If the exactness conditions are satisfied, then

$$\frac{dF(x, y(x))}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = M(x, y) + N(x, y)\frac{dy}{dx},$$

which is just the left-hand-side of the equation above. Hence the equation can be rewritten as

$$\frac{dF(x, y(x))}{dx} = 0,$$

and that the solutions are obviously:

$$F(x, y) = C,$$

where $C$ is any suitable constant.

## 2. Test for Exactness; Solution Algorithm

Consider the ODE

$$2x + y^3 + 3xy^2\frac{dy}{dx} = 0.$$

If you are given the function

$$F(x, y) = xy^3 + x^2$$

and verified that it satisfies

$$F_x(x, y) = y^3 + 2x, \quad F_y(x, y) = 3xy^2,$$

then you realize that the above equation is actually

$$\frac{dF(x, y(x))}{dx} = 0,$$

i.e., exact; and the solutions look like

$$F(x, y) = xy^3 + x^2 = C,$$

or

$$y = \left(\frac{C - x^2}{x}\right)^{1/3}.$$

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However, given an ODE taking the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0,$$

one could not immediately tell whether it is exact; even if so, what the function $F(x,y)$ is. The good news is, if the equation is exact, i.e., there exists and $F(x,y)$ such that $F_x = M$ and $F_y = N$, then further partial derivation gives

$$F_{yx} = M_y, \quad F_{xy} = N_x.$$

Since the mixed partial derivatives of $F$ must be equal, we have obtained a *necessary* condition for exactness: $M_y = N_x$. Furthermore, it can be shown that this necessary condition is also sufficient, and that one could find the function $F(x,y)$ by direct integration. We summarize these fact in the following

**Proposition**. The first order ODE

$$M(x,y)+N(x,y)\frac{dy}{dx}=0,$$

is exact if and only if the functions $M$ and $N$ satisfy

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

**Example**. The equation

$$e^y + (xe^y + 2y)\frac{dy}{dx} = 0$$

is exact, since

$$(e^y)_y = e^y = (xe^y + 2y)_x.$$

Suppose that $F(x,y)$ satisfies $F_x = e^y$ and $F_y = (xe^y + 2y)$. Direct integration of the first equality gives

$$F(x,y) = xe^y + h(y),$$

where $h(y)$ can be any differentiable function in $y$. Now apply the second equality, and obtain

$$xe^y + 2y = F_y = \frac{\partial}{\partial y}(xe^y + h(y)) = xe^y + h'(y).$$

(Note: The terms involving $x$ are canceled out on both sides, and this happens in general when you're solving an exact ODE in this way. See the proof of the Proposition for a reason.) Therefore,

$$h'(y) = 2y,$$

$$h(y) = y^2 + D,$$

where $D$ is a constant. By the discussion before, the solutions to the ODE are

$$xe^y + y^2 = C,$$

where $C$ is a constant. (Note: In the final step, the integration constants are merged into one constant. Thus, it is not necessary to introduce the constant $D$ above.)

2.1. **Integrating Factors.**
If a first order ODE is not exact, is it possible to obtain an equivalent form of the equation which is exact? One attempt that might work is to see if the equation become exact after being multiplied by a nonzero function. Not surprisingly this technique is called (again) *integrating factors*. Usually, if we multiply the equation $M + Ny' = 0$ by a function of $x, y$, say, $\mu(x, y)$, obtaining

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{dy}{dx} = 0,$$

the exactness condition will impose a PDE on $\mu$, which offers little help. However, restricting to the single-variable cases $\mu = \mu(x)$ or $\mu = \mu(y)$ may not be a bad idea.

**Case 1**. $\mu = \mu(x)$.
The equation

$$\mu(x)M(x,y) + \mu(x)N(x,y)\frac{dy}{dx} = 0$$

is exact if and only if

$$(\mu(x)M)_y = (\mu(x)N)_x,$$

that is

$$\mu(x)M_y = \mu'(x)N + \mu(x)N_x,$$

or, equivalently,

$$\mu'(x) = \frac{M_y - N_x}{N}\mu.$$

This looks like an ODE in $\mu(x)$, but only when

$$\frac{M_y - N_x}{N}$$

is a function of $x$ *alone*! If this is satisfied, then $\mu(x)$ can be found by separation of variables.

**Case 2.** $\mu = \mu(y)$.
In this case, the condition for $\mu$ to exist is that

$$\frac{M_y - N_x}{-M}$$

is a function of $y$ *alone*. If so, then $\mu = \mu(y)$ can be found by separation of variables.

**Example**. The ODE

$$(3xy + y^2) + (x^2 + xy)y' = 0,$$

has an integrating factor $\mu(x)$ since

$$\frac{M_y - N_x}{N} = \frac{3x + 2y - 2x - y}{x^2 + xy} = frac1x$$

is a function of $x$. Thus by

$$\mu'(x) = \frac{M_y - N_x}{N}\mu = \frac{1}{x}\mu,$$

solving by separation of variables gives

$$\mu(x) = x.$$

Hence, the equation

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0$$

is exact. Integrating $F_x(x,y) = M(x,y)$ in $x$ gives

$$F(x,y) = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Further considering $F_y(x,y) = N(x,y)$ gives

$$x^3 + x^2y + h'(y) = x^3 + x^2y,$$

$$\Rightarrow \quad h'(y) = 0.$$

Hence, $h(y)$ is a constant, and solutions of the original ODE are

$$x^3y + \frac{1}{2}x^2y^2 = C.$$