LECTURE 5: AUTONOMOUS EQUATIONS

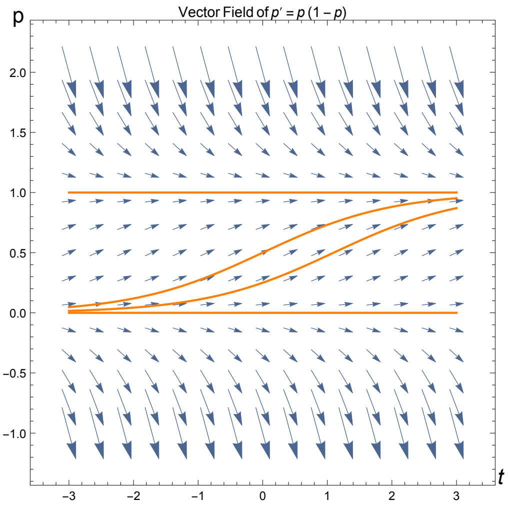
In this lecture, we consider first order equations whose $f(x, y)$-term only depends on the variable y. *Slope fields* and *phase-line plots* are used to study the qualitative behavior of solutions (e.g., equilibrium and stability).

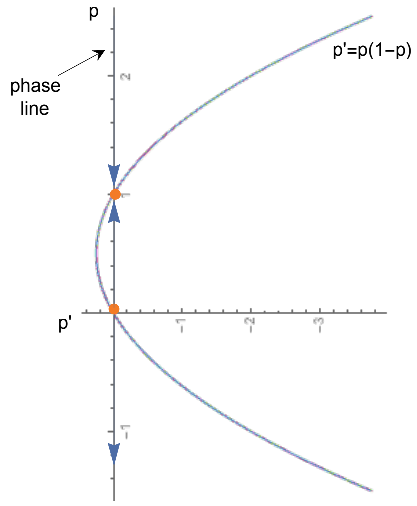
Definition. A first order ODE

$$\frac{dy}{dx} = f(x, y)$$

is said to be **autonomous** if $f(x, y)$ is a function of y *only*.

It is clear that the direction field of an autonomous equation is invariant under translation in the x-direction.





Logistic Model. A plot of the direction field of the standard logistic equation,

$$\dot{p} = p(1 - p),$$

is shown above, together with the graph of the function $g(p) = p(1 - p)$. Note that I have rotated the graph of $g(p) = p(1 - p)$ counter-clockwise by $90°$ so that the p axis goes parallel with that in the vector field plot. In doing this, it is easy to see that the p-intercept of the graph of $g(p) = p(1-p)$ corresponds to the equilibrium solutions (i.e., solutions that are constant functions). Moreover, one could see that when p take values between $(0, 1)$, $\dot{p}$ remains positive. On the direction field side, this means that all the arrows have positive tangent as long as $p \in (0, 1)$. To capture this property, we put a upward arrow between $(0, 1)$ along the p-axis. For a similar reason, we put a downward arrow within the interval $(1, \infty)$ along the p-axis, to indicate that when p takes value in this interval, arrows in the direction field have negative tangent. In addition, we obtain a downward arrow in the interval $(-\infty, 0)$ along the p-axis for the same reason. The p-axis in the graph on the right, with all the arrows and equilibriums marked, is called the **phase line plot** corresponding to the logistic equation $\dot{p} = p(1-p)$. Now, convince yourself that these arrows are capable of telling us the stability of all the equilibriums.

Date: 09/07/16.

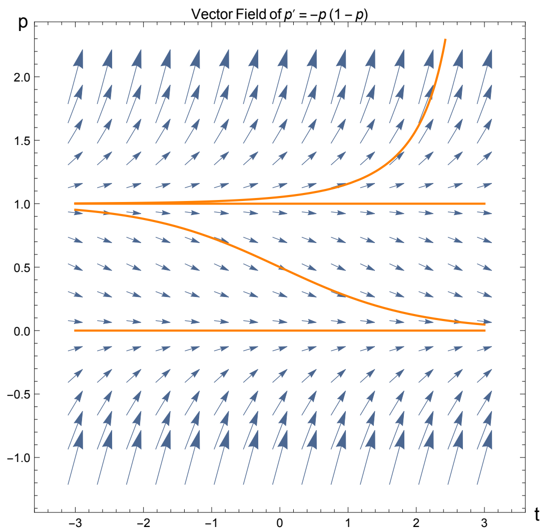
(Do you see that along the phase line, nearby the equilibrium $p = 1$, arrows are pointing towards it? How about $p = 0$? What is your conclusion?)

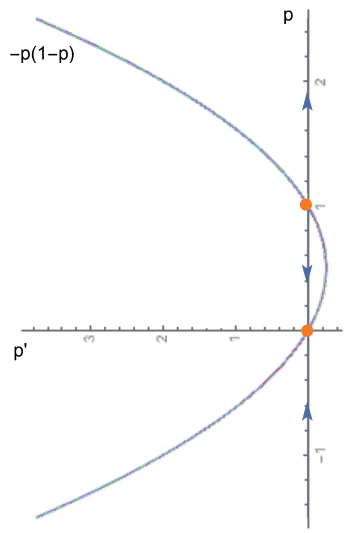
One of the reasons that we like to use phase lines to do analysis is their simplicity: (i) for an autonomous equation in the general form $\dot{y} = g(y)$, the y intercept of the graph of $g(y)$ tells us the equilibria of the system; (ii) by figuring out the sign of $g(y)$ in each interval separated by the equilibrium points, we could draw arrows along these intervals as we did before, and thus be able to tell whether an equilibrium is stable or not.

Threshold Model. Consider the following scenario: for certain species, we do not observe exponential growth in population when the population is below certain level, but rather, the specie would more likely to go to extinction. This scenario can be modeled by the following equation:

$$\dot{p} = -rp\left(1 - \frac{p}{T}\right).$$

You may observe this equation differs from the general logistic equation only by a negative sign on the right hand side. So if we draw the graph of $g(p)$ in this case, it looks exactly like the graph of $g(p)$ in the logistic model flipped over the p-axis. Therefore, when $r = T = 1$, the phase line could be obtained simply by reversing all the arrows in the phase line of the logistic equation. In the plot below, we can see that the equilibrium $p = 0$ is stable and $p = 1$ is not.





Remark. (1) It should be pointed out here that the phase line analysis has its own limitations. For instance, in the standard threshold model with $r = T = 1$, we learned from the phase line that solutions would increase if we set an initial value to be above 1. You may guess that the solution then increases forever (for all x). However, this is not the case. By directly solving the equation using separation of variables, we could see that such solutions goes out of bound in finite time! (Try to verify this as an exercise.) This is something that we could not tell from the phase line alone.

(2) Remembering the bigger picture of modeling, you may have noticed something unrealistic in the threshold model: the specie that our model describes is so "dangerous" that it will either go extinction or reach infinity amount of population, and the population $p = 1$ is unstable. A natural question is: is there a model which takes into

account both the threshold and the environmental capacity? Yep, but let us see if we can construct such a model by ourselves.

Threshold & Environmental Cap: Combination. This time we draw the phase line first, based on what we would expect from the model. First, let's mark the three critical points on the p-axis: 0, T, and R ($T < R$), where T and R are the threshold and capacity. By our argument above, we wish that, for the initial conditions in the intervals $(0,T)$, (T,R), and (R,∞), the solutions respectively descends to zero, increases to R, and decreases to R. This tells us which arrows to put in these intervals: downward for $(0,T)$ and (R,∞), upward for (T,R). Essentially, we are looking for a function $g(p)$ that vanishes at 0, T and R. A natural choice would be

$$g(p) = \lambda p \left(1 - \frac{p}{T}\right)\left(1 - \frac{p}{R}\right),$$

for some constant λ. Now in our case, we know that for $p > R$, $g(p) < 0$. Thus $\lambda < 0$. Let $-\lambda = r$. Therefore, we have obtained the model:

$$\dot{p} = -rp\left(1 - \frac{p}{T}\right)\left(1 - \frac{p}{R}\right).$$

To check that this model works as expected, we plot the stream line of the vector field in the case $r = T = 1$ and $R = 3$ below.