LECTURE 4: EXISTENCE AND UNIQUENESS THEOREM

Given a first order ODE of the form
$$\frac{dy}{dx} = f(x, y),$$
one does not have a general theory for finding its solutions. What's worse, given an initial value, there may not even be a solution to the equation with the specified initial value. We introduce a version of *Existence and Uniqueness theorem* for first order ODEs, which gives *sufficient* conditions for a solution of an initial value problem to exist *locally* (i.e., only nearby the initial point).

1. The Existence and Uniqueness Theorem

Some terminology: we say that there exists a **local solution** for the initial value problem
$$\text{(I)} \quad \begin{cases} y' &= f(x, y) \\ y(x_0) &= y_0 \end{cases}$$
if there exists a differentiable function $f(x, y)$, defined on a (small) open interval containing x_0, say, $I = (x_0 - \epsilon, x_0 + \epsilon)$, so that both the equation and the initial value are satisfied.

Attention may be caught by the term "small open interval". How "small" is small enough? Consider the the first order ODE for the circle in the plane, centered at the origin:
$$y\frac{dy}{dx} = -x.$$
If you choose the origin $(0, 0)$ as the initial condition, there is no circle passing through this point unless the circle degenerates into a point. Therefore, there is no local solution with the initial value $(0, 0)$. For the same reason, there is no local (but only one-sided) solution with the initial value $(1, 0)$, since the x-value of the points on the circle passing through $(1, 0)$ must be ≤ 1. If you choose an initial value off the x-axis, then obviously, there exists a local solution; but the maximum x-interval over which a local solution might extend to remain valid is dependent on the initial value.

Theorem. If, for the initial value problem (I), there exists some rectangle region $R = (x_0 - \delta, x_0 + \delta) \times (y_0 - \tau, y_0 + \tau)$ containing the initial value point (x_0, y_0), on which $f(x, y)$ and $\partial f/\partial y$ are both continuous in x, y, then there exists a local solution of (I) and the solution is unique within its domain of definition.

2. Remarks

1. Revisiting the equation for circles in the example above. The equation can be rewritten as
$$y' = f(x, y) = -\frac{x}{y}.$$

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Clearly, $f(x,y)$ and $\partial f/\partial y(x,y) = x/y^2$ are both continuous away from $\{y = 0\}$. Therefore, for an initial value (x_0, y_0) with $y_0 \neq 0$, the rectangle can be chosen to be an entire half plane (e.g., the upper-half plane if $y_0 > 0$); and a local solution of the initial value problem exists.

2. In the following example, even though there is no restriction for choosing the rectangle R, the domain of definition of a solution is restricted. Consider

$$y' = y^2 + 1.$$

Clearly, both $y^2 + 1$ and $\partial(y^2 + 1)/\partial y = 2y$ are continuous over all of $\mathbb{R}^2$, thus, one could simply choose the rectangle R to be the entire xy-plane. On the other hand, the general solution to the equation looks like

$$y = \tan(x + C).$$

Therefore, it is not possible for a local solution to continuously extend is domain of definition to all over $\mathbb{R}$.

3. To illustrate the uniqueness part of the theorem, consider the initial value problem

$$\begin{cases} y' & = y^{1/3} \\ (x_0, y_0) & = (0,0). \end{cases}$$

There is the obvious solution $y(x) = 0$. Realizing the separability and solving by integration gives another:

$$y(x) = \begin{cases} (2x/3)^{3/2}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Checking the conditions in the initial value problem, we see that $f(x,y) = y^{1/3}$ is continuous everywhere; however, $\partial f/\partial y = y^{-2/3}$ is not even defined at $(0,0)$. On the other hand, as long as $y \neq 0$, both $f(x,y)$ and $\partial f/\partial y$ are continuous. By the Existence and Uniqueness Theorem, for an initial value (x_0, y_0) along the curve

$$y(x) = y^{-2/3}$$

satisfying $(x_0, y_0) \neq (0,0)$, a local solution exists and is unique on its domain of definition. However, extension of its domain of definition is restricted if one wants to retain uniqueness (in this case, cannot extend across $x = 0$).

4. This version of Existence and Uniqueness Theorem is a *sufficient* statement (i.e., condition $P \Rightarrow$ conclusion Q). Necessity (conclusion $Q \Rightarrow$ condition P) is not guaranteed. Thus, it is not surprising to have an initial value problem for which $\partial f/\partial y$ is not defined at the initial point (x_0, y_0) (condition P fails), but has a local unique solution passing through (x_0, y_0) (conclusion Q still holds). As an example, consider

$$\begin{cases} y' & = |y| \\ (x_0, y_0) & = (0,0). \end{cases}$$

Here $\partial|y|/\partial y$ is not defined at any point with y-coordinate being zero. However, existence and uniqueness of solution is guaranteed by a stronger version of the theorem (see *ODE and PDE Notes I*, p.17).