LECTURE 33: NONHOMOGENEOUS HEAT CONDUCTION PROBLEM

1. General Solving Procedure

The general nonhomogeneous 1-dimensional heat conduction problem takes the form

$$\begin{cases} \text{Eq} : [p(x)u_x]_x - q(x)u + F(x,t) = r(x)u_t, \quad 0 \le x \le 1, t > 0, \\ \text{BV} : u_x(0,t) - h_1 u(0,t) = 0, \quad u_x(1,t) - h_2 u(1,t) = 0, \\ \text{IC} : u(x,0) = f(x). \end{cases}$$

A few remarks are in order. First, the nonhomogeneity is due to the term $F(x,t)$ in the equation. Second, the boundary conditions as written may be interpreted as assuming that the rate of heat loss at both ends of the rod is proportional to the temperature there; for example, setting $h_1 = 0$ would mean that the the left end of the rod is insulated.

Direct application of the method of separation of variables does not work here, since the expression of $F(x,t)$ is unknown. Rather, knowing that the underlying homogeneous is relevant to the Sturm-Liouville problem

$$\begin{cases} L[y] = \lambda r(x) y, \quad 0 \le x \le 1, \\ y'(0) - h_1 y(0) = 0, \quad y'(1) - h_2 y(1) = 0 \end{cases}$$

turns out to be useful for finding solutions. For convenience, let λ_k denote the eigenvalues of this Sturm-Liouville problem; ϕ_k the corresponding normalized eigenfunctions.

Now, for each t, view $u(x,t)$ as a function in x. If $u_x(x,t)$ is continuous on $(0,1)$ for each fixed t, then it is possible to obtain a decomposition into ϕ_k's. Such decompositions clearly depend on the variable t. Hence, we may write

$$u(x,t) = \sum_{k=1}^{\infty} b_k(t)\phi_k(x).$$

Written as such, $u(x,t)$ automatically satisfies (BV). One may then ask whether or not it satisfies the nonhomogeneous heat equation.

In fact, we need

$$\left[p(x)\sum_{k=1}^{\infty} b_k(t)\phi_k'(x)\right]_x - q(x)\sum_{k=1}^{\infty} b_k(t)\phi_k(x) + F(x,t) = r(x)\sum_{k=1}^{\infty} b_k'(t)\phi_k(x).$$

Rearranging terms, this is equivalent to

$$\sum_{k=1}^{\infty} b_k(t)L[\phi_k] + \sum_{k=1}^{\infty} b_k'(t)r(x)\phi_k(x) = F(x,t).$$

Using $L[\phi_k] = \lambda_k r(x)\phi_k$ gives

$$r(x)\sum_{k=1}^{\infty}(b_k'(t) + \lambda_k b_k(t))\phi_k(x) = r(x)\frac{F(x,t)}{r(x)}.$$

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One could use the projection formula to obtain

$$\frac{F(x,t)}{r(x)} = \sum_{k=1}^{\infty} c_k(t)\phi_k(x),$$

where

$$c_k(t) = \int_0^1 r(x)\frac{F(x,t)}{r(x)}\phi_k(x)dx = \int_0^1 F(x,t)\phi_k(x)dx.$$

Therefore the condition on $b_k(t)$ further simplifies to

$$\sum_{k=1}^{\infty}(b_k'(t) + \lambda_k b_k(t) - c_k(t))\phi_k(x) = 0,$$

or

$$b_k'(t) + \lambda_k b_k(t) = c_k(t), \quad k = 1, 2, 3, ...$$

Also note that the initial condition $u(x,0)$ requires that

$$u(x,0) = \sum_{k=1}^{\infty} b_k(0)\phi_k(x) = f(x),$$

leading to

$$b_k(0) = \int_0^1 r(x)f(x)\phi_k(x).$$

Thus, finding $b_k(t)$'s amounts to solving the first order initial value problems

$$\begin{cases} b_k'(t) + \lambda_k b_k(t) = c_k(t), \\ b_k(0) = \int_0^1 r(x)f(x)\phi_k(x). \end{cases}$$

It is easy to see that $\mu_k(t) = e^{\lambda_k t}$ are integrating factors. And the solutions are

$$b_k(t) = e^{-\lambda_k t}\int_0^t c_k(s)e^{\lambda_k s}ds + B_k e^{-\lambda_k t},$$

where

$$B_k = \int_0^1 r(x)f(x)\phi_k(x).$$

2. Examples

1. Solve the initial boundary value problem

$$\begin{cases} u_t = u_{xx} + xe^{-t}, \quad 0 < x < a, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(a,t) = 0, \\ u(x,0) = x - \frac{a}{2}. \end{cases}$$

Solution. First notice that, for this problem, the domain of x is not $[0,1]$, hence not exactly in the standard form; but this does not prevent us from using the idea developed in the previous section to solve the problem. The associated homogeneous equation is

$$u_t = u_{xx}.$$

It follows that (by separating the variables) the associated Sturm-Liouville type problem is

$$\begin{cases} X'' + \lambda X = 0, \quad 0 < x < a, \\ X'(0) = X'(a) = 0. \end{cases}$$

By a standard argument, we see that the eigenvalues are

$$\boxed{\lambda_0 = 0, \quad \lambda_k = (k\pi/a)^2, (k = 1, 2, ...),}$$

corresponding to the eigenfunctions

$$\boxed{X_0(x) = \frac{1}{2}, \quad X_k(x) = \cos\left(\frac{k\pi}{a}x\right).}$$

(Note that these X_k's are not normalized. In fact, we do not always need to normalize them, as the following calculation shows.)

If we let

$$\boxed{u(x,t) = \sum_{k=0}^{\infty} b_k(t)X_k(x),}$$

the function u automatically satisfies the boundary condition of the original IBVP. For it to satisfy the equation, we need

$$\frac{\partial}{\partial t}\left(\sum_{k=0}^{\infty} b_k(t)X_k(x)\right) = \frac{\partial^2}{\partial x^2}\left(\sum_{k=0}^{\infty} b_k(t)X_k(x)\right) + xe^{-t},$$

which is

$$\sum_{k=0}^{\infty} b_k'(t)X_k(x) = \sum_{k=0}^{\infty} b_k(t)X_k''(x) + xe^{-t}.$$

Noting that $X_k'' + \lambda X_k = 0$ and supposing that

$$x = \sum_{k=0}^{\infty} c_k X_k(x),$$

we have

$$\sum_{k=0}^{\infty} b_k'(t)X_k(x) = -\sum_{k=0}^{\infty} b_k(t)\lambda_k X_k(x) + \sum_{k=0}^{\infty} c_k e^{-t} X_k(x),$$

or equivalently,

$$\sum_{k=0}^{\infty}(b_k'(t) + \lambda_k b_k(t) - c_k e^{-t})X_k(x) = 0.$$

To solve for $b_k(t)$'s, it suffices to solve the first order ODEs

$$b_k'(t) + \lambda_k b_k(t) = c_k e^{-t}.$$

This is easily solved using integrating factors, getting

$$\boxed{b_k(t) = e^{-\lambda_k t}\int_0^t c_k e^{-s} e^{\lambda_k s} ds + b_k(0)e^{-\lambda_k t},}$$

Of course, the parameters c_k and $b_k(0)$ are still remaining to be computed.

The initial conditions $b_k(0)$ are determined by the equality

$$u(x,0) = \sum_{k=0}^{\infty} b_k(0)X_k(x) = x - \frac{a}{2},$$

and the constants c_k are determined by

$$x = \sum_{k=0}^{\infty} c_k X_k(x).$$

Noting that $\{X_k(x)\} = \{1/2, \cos(\pi x/a), \cos(2\pi x/a)...\}$, $b_k(0)$ and c_k are respectively the Fourier coefficients when $x - \frac{a}{2}$ and x are expanded as a cosine series. We have

$$\boxed{c_0 = \frac{2}{a}\int_0^a x dx = a,}$$

$$\boxed{c_k = \frac{2}{a}\int_0^a x\cos(k\pi x/a)dx = \frac{a^2}{(k\pi)^2}[(-1)^k - 1].}$$

Subtracting $a/2$ from

$$\frac{c_0}{2} + \sum_{k=1}^{\infty} c_k X_k(x)$$

gives

$$\boxed{b_0(0) = 0, \quad b_k(0) = c_k = \frac{a^2}{(k\pi)^2}[(-1)^k - 1], (k \geq 1).}$$

This completes solving the original IBVP.

2. Solve the IBVP

$$\begin{cases} u_t = u_{xx} + 2\cos\frac{\pi x}{L}, & 0 < x < L, \quad t > 0, \\ u_x(0,t) = 0, \quad u_x(L,t) = 0, \\ u(x,0) = x. \end{cases}$$

Solution. The underlying Sturm-Liouville type BVP is

$$\begin{cases} X'' + \lambda X = 0, & 0 < x < L, \\ X'(0) = X'(L) = 0. \end{cases}$$

This leads to the eigenvalues

$$\boxed{\lambda_0 = 0, \quad \lambda_k = (k\pi/L)^2, (k = 1, 2, ...),}$$

and the eigenfunctions

$$\boxed{X_0(x) = \frac{1}{2}, \quad X_k(x) = \cos\left(\frac{k\pi}{L}x\right).}$$

Setting

$$\boxed{u(x,t) = \sum_{k=0}^{\infty} b_k(t) X_k(x),}$$

we have that the boundary conditions in the original IBVP are automatically satisfied. To satisfy the equation, we need

$$\sum_{k=0}^{\infty} b_k'(t) X_k(x) = -\sum_{k=0}^{\infty} b_k(t)\lambda_k X_k(x) + 2X_1(x).$$

This gives rise to the equations

$$b_k'(t) + \lambda_k b_k(t) = 0, \quad k \neq 1,$$

$$b_1'(t) + \lambda_1 b_1(t) = 2.$$

As a result,

$$\boxed{b_k(t) = b_k(0)e^{-\lambda_k t}, \quad k \neq 1}$$

by direct integration, and

$$\boxed{b_1(t) = \frac{2}{\lambda_1} + \left(b_1(0) - \frac{2}{\lambda_1}\right)e^{-\lambda_1 t}}$$

by using an integrating factor.

Now it suffices to find the $b_k(0)$'s. Note that $u(x,0) = x = \sum_{k=0}^{\infty} b_k(0)X_k(x)$, and that, with the choice of $X_k(x)$, $b_k(0)$ are simply the Fourier coefficients when x is expanded as a cosine series of period $2L$. That is,

$$\boxed{b_k(0) = \frac{2}{L}\int_0^L x\cos(k\pi x/L)dx = \begin{cases} \frac{L^2}{(k\pi)^2}[(-1)^k - 1], & k \neq 0, \\ L, & k = 0. \end{cases}}$$

The solution of the original IBVP is obtained simply by putting together the boxed expressions.

The following example[1] is put in an abstract setting, but the idea of solution is still along the line as with the previous two examples.

Example 3. Consider the PDE

$$(t+1)u_t = -L[u], \quad 0 < x < \pi, \quad t > 0,$$

with homogeneous boundary conditions at $x = 0$ and $x = \pi$. Suppose that L is self-adjoint with eigenvalues $\lambda_n = n^2$ ($n = 1, 2, ...$) and eigenfunctions ψ_n; and that these eigenfunctions are complete in the sense that any continuously differentiable function $f(x)$ on $[0,1]$ admits a decomposition $f(x) = \sum b_k\psi_k$ with equality holding on $(0,1)$. Furthermore, we're given the initial condition

$$u(x,0) = 3\psi_1(x) + 4\psi_2(x).$$

What, then, is the solution of this IBVP?

Solution. First write the solution (if exists) as

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\psi_n(x).$$

The boundary conditions are automatically satisfied. For the PDE, we need

$$\sum (t+1)b_n'(t)\psi_n(x) = -\sum n^2 b_n(t)\psi_n(x),$$

leading to the separable first order ODEs:

$$(t+1)b_n'(t) + n^2 b_n(t) = 0.$$

The solutions are

$$b_n(t) = b_n(0)(t+1)^{-n^2}.$$

To determine $b_n(0)$, note that

$$u(x,0) = \sum b_n(0)\psi_n(x) = 3\psi_1(x) + 4\psi_2(x).$$

[1]Adapted from a midterm exam of Fall 2015.

Hence $b_1(0) = 3$, $b_2(0) = 4$ and all other $b_k(0)$'s are zero.

In sum,

$$u(x,t) = 3(t+1)^{-1}\psi_1(x) + 4(t+1)^{-4}\psi_2(x)$$

is the solution of the IBVP.