LECTURE 32: NONHOMOGENEOUS STURM-LIOUVILLE BVP

Recall that the Sturm-Liouville operator L is defined as $L[y] = -[p(x)y']' + q(x)y$. The non-homogeneous Sturm-Liouville BVP is then by definition

$$\text{(nhSL)} \begin{cases} L[y] = \mu r(x)y + f(x), \quad 0 \le x \le 1, \\ \alpha_1 y(0) + \beta_1 y'(0) = 0, \\ \alpha_2 y(1) + \beta_2 y'(1) = 0, \end{cases}$$

where $r(x) > 0$ and μ is a *given* constant. As usual, L is conveniently viewed as an operator on the space of twice differentiable functions defined on $[0,1]$ which satisfy the boundary conditions above. Also notice that the *nonhomogeneity* is only assumed to affect only the equation (but not the boundary conditions), comparing to the homogeneous Sturm-Liouville BVP.

As an example, taking $p(x) = 1$, $q(x) = 0$ in the operator L; $r(x) = 1$, $f(x) = x$ and $\mu = 1$ in the equation, the following BVP

$$\begin{cases} -y'' = y + x, \quad 0 \le x \le 1, \\ y(0) = y(1) = 0 \end{cases}$$

is clearly a nonhomogeneous Sturm-Liouville BVP in the above sense. Approaching as a usual two-point BVP, we first find that the general solution of the equation

$$y'' + y = -x$$

take the form

$$y(x) = -x + c_1 \cos x + c_2 \sin x;$$

then the boundary conditions enforce that

$$c_1 = 0, \quad c_2 = \frac{1}{\sin 1}.$$

Therefore,

$$y(x) = -x + \frac{\sin x}{\sin 1}$$

is the solution of the BVP.

Now suppose that everything in the above example remains unchanged except that $\mu = 2$ now, leading to the BVP

$$\begin{cases} -y'' = 2y + x, \quad 0 \le x \le 1, \\ y(0) = y(1) = 0 \end{cases}$$

To solve this, you'll need to solve a new second order ODE, then use the boundary conditions to determine certain coefficients. This imposes no extra technical difficulty, but there seems to be a better approach to take, particularly in light of the properties of eigen-values/eigen-functions associated to a homogeneous Sturm-Liouville problem.

The underlying homogeneous Sturm-Liouville BVP associated to the problem (nhSL) is:

$$\text{(hSL)} \begin{cases} L[u] = \lambda r(x)u, \quad 0 \le x \le 1, \\ \alpha_1 u(0) + \beta_1 u'(0) = 0, \\ \alpha_2 u(1) + \beta_2 u'(1) = 0. \end{cases}$$

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We have seen that the eigenvalues/eigenfunctions of (hSL) can be indexed by positive integers and any differentiable function on $[0,1]$ admits a decomposition into a combination of the eigenfunctions (and the convergence holds on the open interval $(0,1)$). For convenience, let the eigenvalues be λ_k, the *normalized* eigenfunctions ϕ_k ($k = 1,2,3,...$).

The decomposition property suggests one to write the solution of (nhSL), say $y(x)$, as

$$y(x) = \sum_{k=1}^{\infty} b_k \phi_k.$$

The function y automatically satisfies the boundary conditions in (nhSL). For the equation $L[y] = \mu r(x) y + f(x)$ to hold, we need

$$L\left[\sum_{k=1}^{\infty} b_k \phi_k\right] = \mu r(x) \sum_{k=1}^{\infty} b_k \phi_k + r(x) \frac{f(x)}{r(x)}.$$

The left hand side is, by assuming that the summation and the differential operator L commute,

$$\sum_{k=1}^{\infty} b_k L[\phi_k] = \sum_{k=1}^{\infty} b_k \lambda_k r(x) \phi_k.$$

In the right hand side, $f(x)/r(x)$ admits a decomposition

$$\frac{f(x)}{r(x)} = \sum_{k=1}^{\infty} c_k \phi_k.$$

Thus, the equation becomes

$$\sum_{k=1}^{\infty} b_k \lambda_k r(x) \phi_k = \mu r(x) \sum_{k=1}^{\infty} b_k \phi_k + r(x) \sum_{k=1}^{\infty} c_k \phi_k.$$

Rearranging terms, we obtain

$$\sum_{k=1}^{\infty} (b_k(\lambda_k - \mu) - c_k) r(x) \phi_k(x) = 0.$$

By the projection formula, this implies that

$$b_k(\lambda_k - \mu) = c_k$$

for all $k = 1,2,3,...$

Of course, whether or not the b_k's can be solved for depends on the values of μ and c_k.

- If μ is not an eigenvalue of (hSL), that is, for all k, we have $\mu \neq \lambda_k$; it can be concluded that

  $$b_k = \frac{c_k}{\lambda_k - \mu}.$$

  And the solution of the (nhSL) is given by

  $$y(x) = \sum_{k=1}^{\infty} b_k \phi_k(x).$$

- If $\mu = \lambda_m$ for some m, then there are two possibilities. **One**, if $c_m = 0$, then solutions of (nhSL) exist and there is one-parameter family of them (since b_m is now free and all the b_k's can be solved for using $b_k = c_k/(\lambda_k - \mu)$ for all $k \neq m$). **Two**, if $c_m \neq 0$, the (nhSL) has no solutions.

Example. As an application, we apply the above method to analyse BVPs taking the form

$$\begin{cases} -y'' = \mu y + x, \quad 0 \le x \le 1, \\ y(0) = y(1) = 0, \end{cases}$$

where μ is a given constant.

The associated homogeneous BVP is easily seen to be

$$\begin{cases} u'' + \lambda u = 0, \quad 0 \le x \le 1, \\ u(0) = u(1) = 0, \end{cases}$$

with eigenvalues and normalized eigenfunctions being

$$\lambda_k = (k\pi)^2, \quad \phi_k = \sqrt{2} \sin k\pi x.$$

Now, corresponding notations, $f(x) = x$. Thus, by $r(x)$=1,

$$c_k = \langle x, \phi_k \rangle_{r(x)} = \sqrt{2} \int_0^1 x \sin k\pi x dx = \frac{\sqrt{2}}{k\pi}(-1)^{k+1}.$$

Now, if $\mu \neq \lambda_k$ for all k, we have

$$b_k = \frac{c_k}{\lambda_k - \mu} = (-1)^{k+1} \frac{\sqrt{2}}{k\pi((k\pi)^2 - \mu)},$$

and

$$y(x) = \sum_{k=1}^{\infty} b_k \phi_k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin k\pi x}{k\pi((k\pi)^2 - \mu)}$$

is the solution of the nonhomogeneous BVP. Otherwise, if μ is an eigenvalue of the associated homogeneous BVP, noting that all c_k's are nonzero, we can conclude that there is *no solution* for the nonhomogeneous BVP.

Exercise. Check, by directly solving the equation then applying the boundary conditions, that the BVP

$$\begin{cases} y'' + \pi^2 y = -x, \quad 0 \le x \le 1, \\ y(0) = y(1) = 0 \end{cases}$$

has no solution.