# LECTURE 3: MODELING WITH FIRST ORDER EQUATIONS

## 1. Rate of change

First order ODEs can be used to model a scenario in which the rate of change of the unknown quantity is known. Such equations are usually easy to write down. However, sometimes one need to decide/check (i) What's an appropriate choice of the unknown function; (ii) What assumptions are made leading to a differential equation, for example, the Lambert's law of absorption in model **III** below; (iii) Do the units of the terms in the differential equation agree. Proceeding to examples:

**I**. In a microbial culture, the reproduction rate of certain bacteria can be assumed to be proportional to the total number of bacteria $B(t)$, if the environmental capacity is relatively large. In differential equations, this can be described as

$$\frac{dB}{dt} = rB.$$

One could ask, if it takes $T$ hours for these bacteria to double its population, how much longer will it take for the population to triple (comparing to the initial $B(t_0)$)?

**II**. Imagine that a cup of hot water in room temperature $T_r$ is cooling down. Denote the temperature of the water as $T(t)$. And assume that the rate at which the water cools down is proportional to the difference of temperature between the water and the surrounding air. The water temperature then satisfies the differential equation

$$\frac{dT}{dt} = -k(T - T_r).$$

Let $T(0) = T_0$ be the initial temperature of the water. Treating $k$ as a parameter, the solution of the equation looks like

$$T = (T_0 - T_r)e^{-kt} + T_r.$$

Suppose you measured that after $t_1$ minutes, the temperature of the water is $T_1$ ($T_r < T_1 < T_0$). What's the value of $k$? How much longer does it take the water to cool down to temperature $T_2$ ($T_r < T_2 < T_1 < T_0$)?

**III**. Assume that a beam of sunlight falls vertically on ocean water, and that the ocean water obeys the *Lambert's law of absorption*. That is, the percentage of incident light absorbed by a thin layer of water is proportional to the thickness of the layer. Letting $h$ denote the depth in sea ($h = 0$ for the sea surface), $I$ for the intensity of light, one could establish the differential equation:

$$\frac{dI}{I} = \lambda dh,$$

where the left hand side is the percentage of light absorbed by an infinitesimally thin layer of water, the right hand side a constant multiple of the thickness of the layer. If, at depth $h_1$, the intensity of light is half its intensity at the sea surface, at what depth

*Date*: 09/02/16.

is the intensity 1/8th of that at the sea surface?

**IV**. Mike initially borrowed $B(0) = B_0$ dollars from a bank, and is returning at a rate of $r$ dollars per month. The bank interest rate is $r$. Assume that rates are compounded continuously. The amount Mike owes at time $t$, say, $M(t)$ satisfies the differential equation

$$\frac{dM}{dt} = rM - k.$$

If Mike's plan is to pay off the debt in three years, how large should $k$ be at least?

## 2. Physics & Geometry

2.1. **Leaking Tank.** Suppose that a tank is of height $H$; and the area of cross sections being $A(h)$. Suppose that the tank is initially filled with water. At time $t_0 = 0$, someone digs a small hole of cross area $a$ at the bottom of the tank. How long does it take for the water to drain down to the bottom of the tank?

Of course, there is no question of choosing $h(t)$, the height of water level to the bottom of the tank, as the unknown function. What's the rate of change of $h$? To answer this, one must consider of the *relations* involved in the current physical system. Water level changes because there is loss of water. Thus, at any time $t$, the rate of water loss is related to the rate in which the height of the water level decreases. On the other hand, the rate of water loss can be characterized by the area of the outlet $a$ times the instant velocity $v$ of the outgoing water.

Assume that that energy is conserved for the entirety of water. During a short period of time $\Delta t$, water measured in volume $\Delta V$ and is just about to leave the opening at the bottom. The loss of the potential energy thus equals to $\rho \Delta V h$, which must be equal to the kinetic energy gained. Thus we have

$$\rho g \Delta V h = \frac{1}{2}\rho \Delta V v^2.$$

Hence $v = (2gh)^{1/2}$.

Finally, combining what we've obtained so far, we have the equation:

$$A(h)\frac{dh}{dt} = -a(2gh)^{1/2}.$$

Note that this equation is separable. For $A(h)$ equals some constant, you can try to solve this equation as an exercise.

**Question**. In the setting of leaking tank, we have assumed the hole to be *small*. How does this affect the equation being established?

2.2. **Brachistochrone.** Let $A$ and $B$ ($A$ on the left of $B$ with altitudes $h_A > h_B$) be fixed points in space. Suppose that a slope with no friction is built between $A$ and $B$. What is the shape of the slope (curve) such that the time for an object, released from rest at point $A$, to reach point $B$ is minimized? We considered in class infinitesimal segments linking $A'C'B'$, with $C'$ constrained to a horizontal line between $A'$ and $B'$; the angels between $A'C'$ and the vertical direction being $\alpha_1$, that for $C'B'$ being $\alpha_2$.

We showed the time taken to travel along $A'C'$ with velocity $v_1$, then along $C'B'$ with velocity $v_2$ (eventually, from $A'$ to $B'$) is minimized if $C'$ is positioned in a way such that

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2}.$$

The continuous version of this equality is simply

$$\frac{\sin\alpha}{v} = C,$$

where $\alpha$ is the angle between the tangent of the curve and the vertical direction; $C$ a constant.

If we denote the time-minimizing curve from $A$ to $B$ as $y = y(x)$, where $A$ has coordinates $(x, y) = (0, 0)$, and the $y$-axis is pointing downwards, $x$-axis pointing to the right; then at $(x, y(x))$ along the curve,

$$\sin\alpha = \frac{1}{\sqrt{(y')^2 + 1}},$$

and

$$v = \sqrt{2gy}.$$

Therefore, the condition that $\frac{\sin\alpha}{v}$ being a constant is equivalent to the equation

$$((y')^2 + 1)y = \lambda,$$

where $\lambda$ is some constant.

Now you may follow the instruction in **Problem 32** of [Boyce& DiPrima, p.67] to complete solving the problem. In particular, note that brachistochrone (literally: shortest-time curve) coincides with a cycloid, i.e., the trajectory of a fixed point on a circle when the circle is rolled without slipping along a horizontal line.