LECTURE 28: ONE-DIMENSIONAL WAVE EQUATION: INITIAL-BOUNDARY VALUE PROBLEMS

Pick an elastic string of length L, hold it tight at the both ends, then pull the middle of the string away from its resting position, release it from static, we'll observe an unique motion of the string over time. Mathematically, the condition we have put on the string are exactly the following:

- Fixed ends: $u(0,t) = u(L,t) = 0$.
- Pulling the string from the resting position: $u(x,0) = f(x)$.
- String released from static: $u_t(x,0) = 0$.

Clearly, this is boundary values together with two pieces of initial values (since the equation has the second derivative with respect to t).

More generally, we could consider the following initial-boundary value problem

$$(\mathrm{I})\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x),\ u_t(x,0) = g(x), & 0 \le x \le L. \end{cases}$$

Furthermore, this problem can be *reduced* to two simpler problems, one with zero initial velocity, the other with zero initial value.

$$(\mathrm{II})\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x),\ u_t(x,0) = 0, & 0 \le x \le L. \end{cases}$$

$$(\mathrm{III})\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = 0,\ u_t(x,0) = g(x), & 0 \le x \le L. \end{cases}$$

Exercise. Show that if $u_1(x,t)$ is a solution of (II) and $u_2(x,t)$ is a solution of (III), then $u_1(x,t) + u_2(x,t)$ is a solution of (I), assuming that $f(x), g(x)$ in (I) are the same as those in (II) and (III).

1. Zero Initial Velocity

That is, equation (II). Its solution is similar to that of the heat equations. First consider separation of variables, then use boundary values to narrow down the forms of the solution, finally, consider a superposition of solutions which also satisfies the initial values.

1.1. Separation of Variables.

Let

$$u(x,t) = X(x)T(t).$$

If this solve the wave equation, we must have

$$X(x)T''(t) = c^2 X''(x)T(t),$$

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that is,

$$\frac{X''}{X}(x) = \frac{T''}{c^2T}(t).$$

As before, we know that the value of this expression is a constant, since it does not depend on t, nor x. So, let

$$\frac{X''}{X}(x) = \frac{T''}{c^2T}(t) = -\lambda,$$

for some constant λ . Written in two ODEs, this is just

$$\begin{cases} X'' + \lambda X = 0, \\ T'' + c^2\lambda T = 0. \end{cases}$$

1.2. **Boundary Values.**

First, the fixed ends gives us

$$u(0,t) = u(L,t) = 0,$$

which, under our assumption of the form of solution $u(x,t) = X(x)T(t)$, is just

$$X(0)T(t) = X(L)T(t) = 0,$$

leading to

$$X(0) = X(L) = 0.$$

With the equation

$$X'' + \lambda X = 0,$$

it is an easy exercise of two-point boundary problem to show that the only eigenvalues are

$$\lambda = \left(\frac{n\pi}{L}\right)^2,$$

and correspondingly,

$$X_n(x) = \sin\frac{n\pi}{L}x, \qquad T_n(t) = a_n\cos\frac{n\pi c}{L}t + b_n\sin\frac{n\pi c}{L}t, \quad n = 1, 2, 3, ...$$

where a_n, b_n are arbitrary constants.

1.3. **Initial Values.**

Now we consider the initial values. In particular, if we let $u_n(x,t) = X_n(x)T_n(t)$, then

$$(u_n)_t(x,0) = 0$$

implies

$$T_n'(0) = 0,$$

hence

$$b_n = 0$$

and, up to multiplication by a constant, we can choose $T_n(t) = \cos\frac{n\pi c}{L}t$ and thus

$$u_n(x,t) = \sin\frac{n\pi}{L}x \ \cos\frac{n\pi c}{L}t.$$

Finally, we note that any *superposition* of the u_n's would satisfy the initial-boundary value problem (II), except for one piece of the initial value: $u(x,0) = f(x)$. Therefore, we let

$$u(x,t) = \sum_{n=1}^{\infty} d_n u_n(x,t) = \sum_{n=1}^{\infty} d_n \sin\frac{n\pi}{L}x \ \cos\frac{n\pi c}{L}t.$$

Hence, by the initial value, we have

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi}{L} x.$$

The Fourier coefficients a sine series gives

$$d_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

This completes the solving (II).

2. Zero Initial Position

That is, consider the initial-boundary value problem (III). Before reading the solution below, you may consider trying to solve it as an exercise, as it is quite similar to the preceding case. In fact, nothing is different in the steps of separation of variables and considering the boundary values. There we had

$$X_n(x) = \sin \frac{n\pi}{L} x, \qquad T_n(t) = a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t, \quad n = 1, 2, 3, ...$$

where a_n, b_n are arbitrary constants.

Now the initial value is different. The homogeneous condition is at the position, which requires

$$u(x,0) = 0.$$

Similarly as before, convince yourself that this leads to $a_n = 0$, and, un to multiplication by a constant,

$$T_n(t) = \sin \frac{n\pi c}{L} t.$$

Therefore,we let

$$u_n(x,t) = X_n(x) T_n(t) = \sin \frac{n\pi}{L} x \ \sin \frac{n\pi c}{L} t.$$

And for each $n = 1, 2, 3, ...$, $u_n(x,t)$ satisfies (III) except for the condition $u_t(x,0) = g(x)$.

Finally, we superpose the u_n's, and set

$$u(x,t) = \sum_{n=1}^{\infty} k_n u_n(x,t) = \sum_{n=1}^{\infty} k_n \sin \frac{n\pi}{L} x \ \sin \frac{n\pi c}{L} t.$$

By $u_t(x,0) = g(x)$,

$$\sum_{n=1}^{\infty} \frac{n\pi c}{L} k_n \sin \frac{n\pi}{L} x = g(x).$$

Therefore, the coefficients k_n are determined as

$$\begin{aligned} k_n &= \frac{L}{n\pi c} \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx \\ &= \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L} dx. \end{aligned}$$

This completes solving (III).