LECTURE 27: ONE-DIMENSIONAL WAVE EQUATION: DERIVATION, D'ALEMBERT'S FORMULA

1. Standing and Traveling Waves

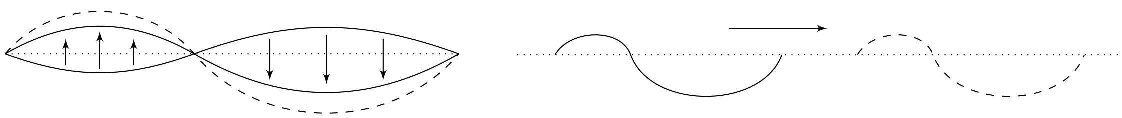
Speaking about waves, there are two basic types: *standing waves* and *traveling waves*. The former, as time evolves, the configuration of the wave changes by an amplifying factor. To be precise, we have

$$u(x,t) = X(x)T(t),$$

in which $u(x,t)$ denote the displacement of the particle at coordinate x of the string at time t from its resting position. The latter, as time evolves, does not change the configuration of the wave, but only travels in a certain direction at a certain velocity. In symbols, this is just

$$u(x,t) = F(x-ct),$$

which means that wave, with configuration being the graph of $F(x)$, is traveling to the right at a velocity of c. Graphically, these two types of waves are illustrated below:



(a) Standing Wave (b) Traveling Wave

Figure 1. Two Types of Waves

For the wave equation that we are about to describe, it turns out that the standing and traveling waves will give us two different views of its solution.

2. The Wave Equation $c^2 u_{xx} = u_{tt}$

Given an elastic string of length L, we could imagine that the mass is concentrated on certain particles along the string (x_n, with $x_{n+1} - x_n = \Delta x$), and that each particle is allowed to move vertically (Figure 2).

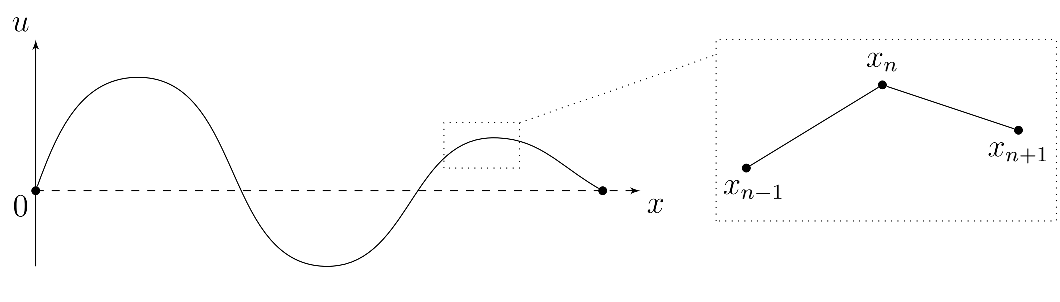


Figure 2. Elastic String

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Furthermore, we assume that the tension along the string is a constant τ. Therefore, the vertical component of the tension of the string connecting to x_n, left and right, are respectively

$$F_\ell = -\tau \frac{u(x_n,t) - u(x_{n-1},t)}{\Delta x}, \qquad F_r = \tau \frac{u(x_{n+1},t) - u(x_n,t)}{\Delta x}.$$

Hence, ignoring gravity, the force of the string at the particle x_n is

$$F = F_\ell + F_r = \tau \frac{u(x_n+\Delta x,t) - 2u(x_n,t) + u(x_n - \Delta x,t)}{\Delta x}.$$

By Newton's second law, we have

$$F = ma = \Delta x \rho u_{tt}(x_n,t),$$

where ρ is the density of the string. This gives us

$$u_{tt}(x_n,t) = \frac{\tau}{\rho} \frac{u(x_n+\Delta x,t) - 2u(x_n,t) + u(x_n - \Delta x,t)}{\Delta x^2}.$$

Taking the limit $\Delta x \to 0$ and using the Taylor expansion at $x = x_n$ (or using the L'Hôpital's rule twice), we obtain the equality

$$u_{tt}(x_n,t) = \frac{\tau}{\rho} u_{xx}(x_n,t).$$

Indeed, there is nothing special about the position of x_n here, so let it be denoted by x instead $(0 < x < L)$. Also let us call τ/ρ as c^2. The wave equation follows:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \qquad 0 < x < L, t > 0.$$

One may ask: what is c? In section 4, we will see that the solution of this equation is the superposition of two traveling waves, one to the left, the other to the right, both with velocity c.

3. Traveling Waves and D'Alembert's Formula

In this section, we look at the solutions of the wave equation

$$c^2 u_{xx} = u_{tt}$$

from a different angle. First, let us consider the new variables

$$\xi = x - ct, \qquad \eta = x + ct.$$

Exercise. Show that under the new variables, the wave equation above is equivalent to

$$u_{\xi\eta} = 0.$$

By the fundamental theorem of calculus (use it twice!), we know that any solution of this equation must be of the form

$$u(\xi,\eta) = F(\xi) + G(\eta),$$

for some functions F, G. (And, in fact, for any differentiable functions $F(x)$ and $G(x)$, $F(\xi) + G(\eta)$ solves the wave equation.)

Therefore, change the variables back to x, t in the solution, we have that any solution of the wave equation must be of the form

$$u(x,t) = F(x - ct) + G(x + ct),$$

which is simply the superposition of two *traveling waves*, one going to the left, the other to the right, both of velocity c.

Furthermore, if we also know initial conditions, say,

$$u(x,0) = f(x), \quad u_t(x,0) = g(x),$$

then the solution is given by the **d'Alembert's Formula**:

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau.$$