LECTURE 24: THE ONE-DIMENSIONAL HEAT EQUATION

The Heat Equation $\alpha^2 u_{xx} = u_t$

Think of a rod with the following data:

- *Length*: L;
- *Distribution of temperature along the rod*: $u(x,t)$, depending on time;
- *Area of cross section*: A, a small number so that the temperature is viewed as a constant when the x coordinate is fixed.

We also need three postulates:

- The rate of heat conduction crossing a unit area is proportional to $\frac{\partial u}{\partial \mathbf{n}}$, where $\mathbf{n}$ is the inward unit normal of the area. In our case, letting ΔH denote the heat transferred past (to the right) the cross section $x = x_0$, we have
$$\Delta H = -\lambda A \frac{\partial u}{\partial x}(x_0, t)\Delta t,$$
for some constant $\lambda > 0$.
- Fixing a short segment in the rod, with volume $\Delta V = A \cdot \Delta x$, the increase in temperature times volume in this segment is proportional to the increase in heat in this segment.
- No heat transfer through the "horizontal" boundary of the rod.

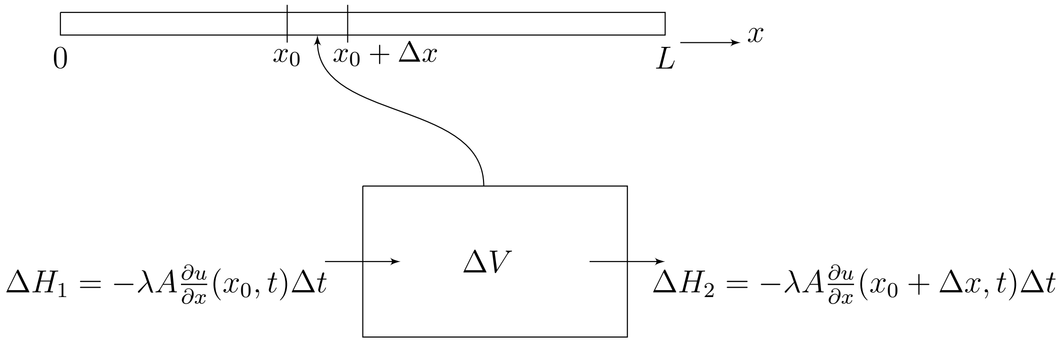

Figure 1. Heat Conduction in a Rod

Now, focus on a segment in the rod shown in the figure above. Within Δt time, ΔV takes in heat from its left face by the amount ΔH_1 and loses heat from its right face by the amount ΔH_2. Hence, the heat in ΔV is increased by
$$\Delta H_1 - \Delta H_2 = \lambda A \Delta t \Big(\frac{\partial u}{\partial x}(x_0 + \Delta x, t) - \frac{\partial u}{\partial x}(x_0, t) \Big).$$
By the second postulate, this increase in heat is proportional to $\Delta u(x_0, t) \cdot \Delta V$, say,
$$\Delta H_1 - \Delta H_2 = \mu \Delta u(x_0, t) \cdot \Delta V = \mu \frac{\partial u}{\partial t}(x_0, t) \Delta t A \Delta x.$$

Date: 10/28/16.

Therefore, we have the relation:

$$\lambda A\Delta t\Big(\frac{\partial u}{\partial x}(x_0+\Delta x,t)-\frac{\partial u}{\partial x}(x_0,t)\Big)=\mu\frac{\partial u}{\partial t}(x_0,t)A\Delta t\Delta x,$$

i.e.,

$$\frac{\lambda}{\mu}\frac{1}{\Delta x}\Big(\frac{\partial u}{\partial x}(x_0+\Delta x,t)-\frac{\partial u}{\partial x}(x_0,t)\Big)=\frac{\partial u}{\partial t}(x_0,t).$$

Passing to the limit $\Delta x\to 0$ gives

$$\frac{\lambda}{\mu}\frac{\partial^2 u}{\partial x^2}(x_0,t)=\frac{\partial u}{\partial t}(x_0,t).$$

Letting $\frac{\lambda}{\mu}=\alpha^2$, and noting that the equality above holds for all $0<x_0<L$, we have thus obtained

$$\alpha^2 u_{xx}(x,t)=u_t(x,t),\qquad (0<x<L, t>0),$$

the equation of heat conduction in a rod.