LECTURE 23: FOURIER CONVERGENCE THEOREM, EVEN AND ODD FUNCTIONS

1. Fourier convergence Theorem

Theorem. If $f(x), f'(x)$ are piecewise continuous on $[-L, L)$, periodic with period $2L$, then its Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right)$$

converges to

$$\begin{cases} f(x), & \text{at the values of } x \text{ where } f(x) \text{ is continuous,} \\ (f(x^+) + f(x^-))/2, & \text{at the values of } x \text{ where } f(x) \text{ is discontinuous.} \end{cases}$$

Example. Let

$$f(x) = \begin{cases} 0, & -L < x < 0, \\ L, & 0 < x < L. \end{cases} \qquad f(x + 2L) = f(x).$$

The Fourier series of $f(x)$, by the theorem, converges to the function

$$\tilde{f}(x) = \begin{cases} 0, & -L < x < 0, \\ L, & 0 < x < L, \\ \frac{1}{2}L, & x = 0. \end{cases} \qquad \tilde{f}(x + 2L) = \tilde{f}(x).$$

Particularly, just by calculating the Fourier series and using the Fourier convergence theorem, we are able to obtain interesting results about the limit of certain infinite series. For instance, if we calculate the Fourier series of the function $f(x)$ in the example above, we get

$$\begin{aligned}
a_0 &= \frac{1}{L} \int_{-L}^{L} f(x) dx = L, \\
a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx = \int_0^L \cos \frac{n\pi x}{L} dx = 0, (n = 1, 2, 3, ...) \\
b_m &= \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{m\pi x}{L} dx = \int_0^L \sin \frac{m\pi x}{L} dx = \frac{L}{m\pi}[1 - (-1)^m], (m = 1, 2, 3, ...) \\
&= \frac{2L}{(2k+1)\pi}, (m = 2k + 1, k = 0, 1, 2, ...).
\end{aligned}$$

Therefore,

$$f(x) \sim \frac{L}{2} + \sum_{k=0}^{\infty} \frac{2L}{(2k+1)\pi} \sin \frac{(2k+1)\pi x}{L}.$$

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Now, consider $x = \frac{L}{2}$, note that this is a point at which $f(x)$ is continuous, hence the Fourier series evaluated at this point converges to L, that is

$$\frac{L}{2} + \sum_{k=0}^{\infty} \frac{2L}{(2k+1)\pi}(-1)^k = L.$$

This is just

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

2. Even and Odd Functions

This is a topic we are familiar with already, at least in terms of the symmetry of the graph of an even or odd function. We will have a review of this topic here since it will enable us to

- halve the effort of calculation of the coefficients in a Fourier series;
- extend a function appropriately so that the result has a desired type of Fourier series.

Definition. Function $f(x)$ is said to be *even* if $f(x) = f(-x)$ for all x, and *odd* if $f(x) = -f(-x)$ for all x.

By the definition, it is easy to see that the sum/difference/product of two even functions is even, the sum/difference of two odd functions is odd, the product of two odd functions is even, the product of an even function and an odd function is odd, etc.

Also, we have the following integral identities:

- $f(x)$ is even:
$$\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx;$$
- $f(x)$ is odd:
$$\int_{-L}^{L} f(x)dx = 0.$$

Applying to Fourier series, we assume that both f, f' are piecewise continuous and periodic with period $2L$, and we have

- $f(x)$ is even:
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}dx, (n = 0, 1, 2, ...)$$
$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = 0.(m = 1, 2, ...)$$
- $f(x)$ is odd:
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = 0, (n = 0, 1, 2, ...)$$
$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{m\pi x}{L}dx.(m = 1, 2, ...)$$

Observe that when $f(x)$ is even, its Fourier series consists only of the cosine terms, and we call it a *cosine series*. Similarly, when $f(x)$ is odd, the Fourier series is called a *sine series*.

Example. The function

$$f(x) = x, \qquad -L \leq x < L, f(x+2L) = f(x)$$

is odd and its Fourier series is a sine series. The function

$$f(x) = |x|, \qquad -L \leq x < L, f(x+2L) = f(x)$$

is even and its Fourier series is an cosine series.

Looking closer at the two functions in the above example, we could see that the value of the functions are the same on the interval $[0, L]$. In other words, $f(x) = x$ on $[0, L]$ can be expanded as a sine series, a cosine series, or a mixture of both. In fact, this is no surprise since in the two cases, we have *extended* the function $f(x) = x$ either oddly or evenly to a periodic function with period $2L$. This inspires us the following: If a function is defined on $[0, L]$ and we need to expand it as a sine(cosine) series of period $2L$, we simply *extend the function oddly(evenly) to* $[-L, L)$ *and periodically to* $(-\infty, \infty)$ *with period* $2L$*, then calculate the Fourier series.*