LECTURE 22: FOURIER SERIES

An intuitive model of the Fourier series is perhaps the sound that we hear: a superposition of vibrations of various amplitudes and frequencies, each represented by either a sine or cosine wave. The formal definition of the *Fourier series* is the following:

$$\frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right),$$

where $a_n (n = 0, 1, 2, ...)$ and $b_n (n = 1, 2, 3, ...)$ are constants. Certainly, when we encounter an infinite series like this, a natural question to ask is its convergence. We will discuss this in the next lecture. For now, we are going to study the Fourier series formally and see what is special about it.

1. Periodic Functions

Looking at the terms in the Fourier series defined above, we observe that all the cosine and sine functions share the *period* $2L$.

As a reminder, a function $f(x)$ is said to be *periodic* with period $T > 0$ if $f(x + T) = f(x)$ for all x. By this definition, it is easy to see that the period of $f(x)$, if exists, is not unique. In fact, if $T > 0$ is a period, then $kT (k = 1, 2, 3, ...)$ are all periods of $f(x)$. The *fundamental period* of a periodic function $f(x)$ is defined as the *smallest* period of $f(x)$.

Useful facts about the periodic functions include: If $f(x)$ and $g(x)$ have the same period T, then $af(x) + bg(x)$ and $f(x)g(x)$, where a, b are constants, all have the period T.

2. L^2-inner product and Euler-Fourier Formula

Let $f(x), g(x)$ be piecewise continuous functions defined on the interval $[\alpha, \beta]$, then the *L^2-inner product* of $f(x), g(x)$ over the interval $[\alpha, \beta]$ is defined as

$$\langle f, g \rangle_{L^2[\alpha,\beta]} = \int_{\alpha}^{\beta} f(x)g(x)dx.$$

When there is no confusion of the notation, we could simply write $\langle f, g \rangle_{L^2[\alpha,\beta]}$ as $\langle f, g \rangle$. Note that the L^2-inner product defined as this satisfies all the axioms of an inner product:

- Symmetry: $\langle f, g \rangle = \langle g, f \rangle$.
- Bilinearity: $\langle af + bg, h \rangle = a\langle f, h \rangle + b\langle g, h \rangle$, a, b constants.
- Positive definity: $\langle f, f \rangle \geq 0$, and $\langle f, f \rangle = 0$ if and only if $f = 0$ almost everywhere.

Geometrically, whenever we have an inner product, there are the notions of *angle* and *length*. In particular, in the case of L^2-inner product, f, g are said to be *orthogonal* if

$$\langle f, g \rangle = 0,$$

and the *length* of f is defined as

$$\|f\|_{L^2} = \langle f, f \rangle^{\frac{1}{2}}.$$

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Returning to Fourier series, note that the Fourier series is an infinite constant coefficient linear combination of the list of functions

$$\{1, \cos\frac{\pi x}{L}, \sin\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \sin\frac{2\pi x}{L}, ...\}$$

Considering the L^2-inner products of these functions over the interval $[-L, L]$, we have the following two claims:

Claim 1. The functions in the list above are mutually orthogonal.

Claim 2. The square of the lengths of all the functions in the list above are L, except that of the function 1, which is $\|1\|^2 = 2L$.

Before proving these claims, we show how they may be useful. Suppose that the Fourier series

$$\frac{a_0}{2} + \sum_{m=1}^{\infty}\Big(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\Big),$$

converges to a function $f(x)$ on the interval $[-L, L]$. By the claims we must have

$$\begin{aligned}
\langle f, 1\rangle &= \langle\frac{a_0}{2} + \sum_{m=1}^{\infty}\Big(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\Big), 1\rangle = \langle\frac{a_0}{2}, 1\rangle \\
&= \frac{a_0}{2}\langle 1, 1\rangle = a_0 L, \\
\langle f, \cos\frac{m\pi x}{L}\rangle &= \langle\frac{a_0}{2} + \sum_{m=1}^{\infty}\Big(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\Big), \cos\frac{m\pi x}{L}\rangle \\
&= a_m\langle\cos\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}\rangle = a_m L, \qquad m = 1, 2, 3, ... \\
\langle f, \sin\frac{m\pi x}{L}\rangle &= \langle\frac{a_0}{2} + \sum_{m=1}^{\infty}\Big(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\Big), \sin\frac{m\pi x}{L}\rangle \\
&= b_m\langle\sin\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}\rangle = b_m L. \qquad m = 1, 2, 3, ...
\end{aligned}$$

Therefore, we have obtained the *Euler-Fourier formula* for calculating the coefficients in a Fourier series:

$$\begin{aligned}
a_n &= \frac{1}{L}\langle f, \cos\frac{n\pi x}{L}\rangle, \qquad n = 0, 1, 2, ... \\
b_m &= \frac{1}{L}\langle f, \sin\frac{n\pi x}{L}\rangle, \qquad m = 1, 2, 3, ...
\end{aligned}$$

Remark. In the argument which leads to the Euler-Fourier formula, we have assumed that the Fourier series converges to $f(x)$. Now, with the formulas, we could ask for the converse: given $f(x)$ defined on $[-L, L]$ with period $2L$, we could find the coefficients a_n, b_m by the formula, hence a Fourier series associated to $f(x)$. Right now, we do not know whether this Fourier series converges to $f(x)$ or not, so we use the symbol $\sim$ to denote their relation:

$$f(x) \sim \frac{a_0}{2} + \sum_{m=1}^{\infty}\Big(a_m\cos\frac{m\pi x}{L} + b_m\sin\frac{m\pi x}{L}\Big),$$

where all the coefficients in the series are calculated by the Euler-Fourier formulas.

Example. Find the Fourier series for the function

$$f(x) = \begin{cases} -x, & -2 \leq x < 0, \\ x, & 0 \leq x < 2. \end{cases} \qquad f(x+4) = f(x).$$

Evidently, this function has period $T = 4$, hence $L = 2$. By the Euler-Fourier formulas,

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)dx = 2; \\ a_m &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{m\pi x}{2}dx = \int_{0}^{2} x\cos\frac{m\pi x}{2}dx \\ &= \frac{-4(1-(-1^m))}{(m\pi)^2} \\ b_m &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{m\pi x}{2}dx = 0. \end{aligned}$$

The Fourier series follows.

Note: We used the relation between even and odd functions and integrals to simplify the evaluation of the integrals above. More details will be discussed in the next lecture.

3. Proof of the Claims

Proof of Claim 1. To check the orthogonality of these functions, it suffices to check that the five types of expressions:

$$\begin{aligned} &\langle 1, \cos\frac{n\pi x}{L}\rangle (n > 0), \quad \langle 1, \sin\frac{n\pi x}{L}\rangle (n > 0), \\ &\langle \cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}\rangle (n \neq m), \quad \langle \sin\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}\rangle (n \neq m), \\ &\langle \cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}\rangle (m, n > 0) \end{aligned}$$

are all zero. Noting that each of these expressions is essentially an integral over the interval $[-L, L]$, by the oddness of the product of the two functions being taken inner product, we know immediately that the second and the fifth types of expressions are always zero.

To check the first type, we have

$$\langle 1, \cos\frac{n\pi x}{L}\rangle = \int_{-L}^{L} \cos\frac{n\pi x}{L}dx = \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_{-L}^{L} = 0.$$

Next, we check that the third type of expressions are equal to zero and leave the fourth one, which is completely analogous, as an exercise:

$$\begin{aligned}
\langle \cos\frac{n\pi x}{L}, \cos\frac{m\pi x}{L}\rangle &= \int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx \\
&= \frac{1}{2}\int_{-L}^{L}\Big(\cos\frac{(m-n)\pi x}{L} + \cos\frac{(m+n)\pi x}{L}\Big)dx \\
&= \frac{L}{2(m-n)\pi}\sin\frac{(m-n)\pi x}{L}\Big|_{-L}^{L} + \frac{L}{2(m+n)\pi}\sin\frac{(m+n)\pi x}{L}\Big|_{-L}^{L} \\
&= 0.
\end{aligned}$$

This completes the proof.

Proof of Claim 2. To check the *square-of-length* statement, we need to calculate

$$\langle 1,1\rangle, \quad \langle \cos\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}\rangle (m=1,2,...), \quad \langle \sin\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}\rangle (m=1,2,...).$$

Now,

$$\|1\|^2 = \langle 1,1\rangle = \int_{-L}^{L} 1dx = 2L.$$

$$\begin{aligned}
\Big\|\cos\frac{m\pi x}{L}\Big\|^2 &= \langle \cos\frac{m\pi x}{L}, \cos\frac{m\pi x}{L}\rangle = \int_{-L}^{L}\cos^2\frac{m\pi x}{L}dx \\
&= \frac{1}{2}\int_{-L}^{L}\Big(\cos\frac{2m\pi x}{L} + 1\Big)dx \\
&= \frac{L}{4m\pi}\sin\frac{2m\pi x}{L}\Big|_{-L}^{L} + L \\
&= L.
\end{aligned}$$

$$\begin{aligned}
\Big\|\sin\frac{m\pi x}{L}\Big\|^2 &= \langle \sin\frac{m\pi x}{L}, \sin\frac{m\pi x}{L}\rangle = \int_{-L}^{L}\sin^2\frac{m\pi x}{L}dx \\
&= \frac{1}{2}\int_{-L}^{L}\Big(1 - \cos\frac{2m\pi x}{L}\Big)dx \\
&= L - \frac{L}{4m\pi}\sin\frac{2m\pi x}{L}\Big|_{-L}^{L} \\
&= L.
\end{aligned}$$

This completes the proof.