LECTURE 20: REVIEW OF LAPLACE TRANSFORMS

This lecture is dedicated to a review of the method of Laplace Transforms consisting mainly of examples and exercises. To start with, let us remind ourselves that only certain *appropriate* functions are "Laplace Transformable" and one such class of functions are characterized by *piecewise continuity* and *of exponential order.* (Note: The impulse functions seem to be outside this class, but their Laplace transforms are defined.) The following exercise is relevant:

Exercise 1: Which of the following functions are *appropriate* for the Laplace transform. For those that are appropriate, find the Laplace transform.

$$(a)\ e^{t^2}, \qquad (b)\ f(t) = \begin{cases} 0, & t \text{ is even,} \\ 1, & \text{otherwise.} \end{cases} \qquad (c)\ e^{2t}.$$

Note: The set of even numbers is a *negligible set* in the sense of integral.

Solution. First recall that *appropriate* means piecewise continuous on $[0, \infty)$ and of the exponential order. In particular, there exist positive constants M, α such that

$$|f(t)| \le Me^{\alpha t}, \qquad t \ge 0.$$

In the list,

(a) e^{t^2} is not of the exponential order because for any $M, \alpha > 0$, $Me^{\alpha t} = e^{(\ln M)\alpha t}$ and $t^2 > (\ln M)\alpha t$ as long as $t > (\ln M)\alpha$. Hence, e^{t^2} is not appropriate.

(b) $f(t) = \begin{cases} 0, & t \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$ is appropriate since it is piecewise continuous and bounded, hence of the exponential order. Now, since the set of even numbers is negligible, the Laplace transform of $f(t)$ must be the same as that of the function $g(t) = 1$. Therefore,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0.$$

(c) e^{2t} is appropriate.

$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \qquad s > 2.$$

The following exercise is an example in which the application of Laplace transform is not restricted to *differential equations*, but may include some *integral equations.*

Exercise 2: Use the Laplace transform to find the function $f(t)$ which satisfies the integral equation

$$f(t) = t + \int_0^t f(x)\sin(t-x)dx.$$

Solution. Observe that the integral equation involves a convolution integral and thus can be rewritten as

$$f(t) = t + \sin t * f(t).$$

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Now, apply the Laplace transform to both sides of this equality, then use the convolution formula, we have

$$F(s) = \frac{1}{s^2} + \frac{1}{1+s^2}F(s).$$

This is just

$$F(s) = \frac{1+s^2}{s^4} = \frac{1}{s^4} + \frac{1}{s^2}.$$

The inverse Laplace transform of $F(s)$ is now evident:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{6}t^3 + t.$$

In class, we have explained why $\delta(t)$ is in a sense the *identity element* with respect to the convolution of two functions. Relevantly, we have the following useful result:
Example 1: Let $\delta_{t_0}(t)$ denote $\delta(t - t_0)$ $(t_0 > 0)$, and let $f(t)$ be a function which is appropriate for Laplce transform. We have by definition

$$(f * \delta_{t_0})(t) = \int_0^t f(t-\tau)\delta(\tau - t_0)d\tau.$$

(1) If $t < t_0$, the integrand $f(t-\tau)\delta(\tau - t_0)$ is then identically zero for $z \in [0, t]$, hence the integral is zero for such values of t. Namely,

$$(f * \delta_{t_0})(t) = 0, \quad t \in [0, t_0).$$

(2) If $t > t_0$, then, because $\tau \in [0, t]$ and this interval contains t_0 as an inner point, we have

$$(f * \delta_{t_0})(t) = \int_0^t f(t-\tau)\delta(\tau - t_0)d\tau = f(t - t_0).$$

Up to the ambiguity of the function value at $t = t_0$, one could put

$$(f * \delta_{t_0})(t) = u_{t_0}(t)f(t - t_0), \quad t_0 > 0.$$

Here is a relevant
Exercise 3: Solve the following initial value problem using Laplace transform

$$y'' + y = 2\delta(t-4), \quad y(0) = y'(0) = 0.$$

Solution. Applying Laplace transform to both sides of the equation gives,

$$(s^2+1)Y(s) = \mathcal{L}\{2\delta(t-4)\}.$$

As a result of the facts $\mathcal{L}(f(t) * g(t)) = F(s)G(s)$ and the previous example, we have

$$\begin{aligned} y(t) &= \sin(t) * (2\delta(t-4)) \\ &= 2u_4(t)\sin(t-4). \end{aligned}$$

When applying the shifting formulas, it is important to identify correctly the functions $f(t)$ in those formulas. For example, one realizes that

$$\mathcal{L}\{u_3(t)(t+1)\}$$

may be computed using the second shift formula $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-sc}F(s)$. Here the constant c can be identified to be $c = 3$, hence $f(t-3) = t+1$ and $f(t) = t+4$. Therefore,

$$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2} + \frac{4}{s}$$

and

$$\mathcal{L}\{u_3(t)(t+1)\} = e^{-3s}\left(\frac{1}{s^2} + \frac{4}{s}\right).$$

Sometimes, it may be useful to introduce intermediate functions. For example, in calculating

$$\mathcal{L}\{u_3(t)e^{-(t-3)}\sin(t-3)\}$$

we notice that $u_3(t)e^{-(t-3)}\sin(t-3)$ is in the form $u_3(t)f(t-3)$ where

$$f(t) = e^{-t}\sin(t).$$

Thus, the second shifting formula implies that

$$\mathcal{L}\{u_3(t)e^{-(t-3)}\sin(t-3)\} = e^{-3s}F(s).$$

But $F(s)$ is in turn computed by the first shift formula:

$$F(s) = \mathcal{L}\{f(t)\} = \{\mathcal{L}e^{-t}\sin(t)\} = \frac{1}{(s+1)^2+1}.$$

Therefore,

$$\mathcal{L}\{u_3(t)e^{-(t-3)}\sin(t-3)\} = e^{-3s}\frac{1}{(s+1)^2+1}.$$

The method of partial fraction decomposition naturally appears in solution of constant coefficient ODEs by Laplace transform. The following exercise is from your most recent homework (#6):

Exercise 4: Use the Laplace and inverse Laplace transforms to solve the second order differential equation

$$y'' - (a+b)y' + (ab)y = 0$$

with initial values set to be $y(0), y'(0)$, where a, b are arbitrary real constants.

Solution. If you are aware of the theory of second order constant coefficient homogeneous ODEs, it is natural to expect to discuss in two cases: $a \neq b$ and $a = b$. For convenience, I denote $y(0)$ as p and $y'(0)$ as q. Laplace transforming both sides of the equation yields

$$(s^2Y - q - ps) - (a+b)(sY - p) + (ab)Y = 0.$$

As a result,

$$Y(s) = \frac{q + ps - (a+b)p}{s^2 - (a+b)s + ab} = \frac{ps + q - (a+b)p}{(s-a)(s-b)} = \frac{p}{s-b} + \frac{q-bp}{(s-a)(s-b)}.$$

Case 1: $a \neq b$. We have

$$\frac{1}{(s-a)(s-b)} = \frac{1}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right),$$

Hence,

$$Y(s) = \frac{p}{s-b} + \frac{q-bp}{a-b}\left(\frac{1}{s-a} - \frac{1}{s-b}\right).$$

The invserse Laplace transform can be carried out, by linearity and the first shift formula, to give

$$y(t) = pe^{bt} + \frac{q - bp}{a - b}(e^{at} - e^{bt}) = \frac{q - bp}{a - b}e^{at} + \frac{pa - q}{a - b}e^{bt}.$$

Case 2: $a = b$. We have

$$Y(s) = \frac{p}{s - b} + \frac{q - bp}{(s - a)^2}.$$

Noting that $\mathcal{L}\{t\} = s^{-2}$ and by the first shift formula, one obtains

$$\mathcal{L}^{-1}\{(s - a)^2\} = e^{at} \cdot t.$$

Hence,

$$y(t) = pe^{bt} + (q - bp)te^{at}.$$