LECTURE 2: SEPARABLE EQUATIONS, HOMOGENEOUS EQUATIONS

1. Separable Equations

A first order ODE with independent variable x

$$\frac{dy}{dx} = f(x, y)$$

is said to be *separable* if the function $f(x, y)$ can be written as

$$f(x, y) = g(x)h(y).$$

A separable equation can be solved by separating the variables and integrating both sides of the equation. The solution is usually an implicit formula for y, i.e., a relation between x, y given by some $F(x, y) = 0$ rather than the explicit solution $y = F(x)$. Proceeding to examples:

Example 1. The equation

$$x + y\frac{dy}{dx} = 0$$

is separable, since the variable and the unknown function can be obviously separated to give

$$xdx = -ydy.$$

Integrating both sides, one obtains

$$\frac{1}{2}x^2 = -\frac{1}{2}y^2 + C,$$

or, equivalently (after renaming the constant C),

$$x^2 + y^2 = C.$$

For $C > 0$, this is just the (algebraic) equation for circles centered at the origin. The implicitness is clear. For example, the solution passing through $(1, 0)$ is the unit circle in $\mathbb{R}^2$ and nearby $(1, 0)$, y is not expressible as a function of x. (However, it is possible to regard x as a function of y near $(1, 0)$.)

Example 2. Similarly, the equation

$$\frac{dy}{dx} - \frac{x^2 + 1}{\sin y + 1} = 0$$

is separable. Integrating both sides of

$$(\sin y + 1)dy = (x^2 + 1)dx$$

gives

$$-\cos y + y = \frac{1}{3}x^3 + x + C.$$

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Therefore, the (implicit) solutions are

$$\frac{1}{3}x^3 + x + \cos y - y = C.$$

Example 3. The population model with the reproduction rate r and environmental capacity C both set to 1 is written:

$$\frac{dp}{dt} = p(1-p).$$

This is separable equation. Separating the variables gives

$$\frac{1}{p(1-p)}dp = dt.$$

The left hand side can be written in a form whose integral is more recognizable:

$$\frac{1}{p(1-p)}dp = \frac{1}{p}dp - \frac{1}{p-1}dp.$$

Integration on both sides of $\frac{1}{p(1-p)}dp = dt$ then gives

$$\ln|p| - \ln|p-1| = t + C.$$

Equivalently,

$$\frac{p}{|1-p|} = Ce^t, \quad (C > 0).$$

Of course, during the integration, one assumes $p, p-1 \neq 0$, thus forgetting the two obvious solutions $p(t) = 0$ and $p(t) = 1$, which should be considered in the final answer:

$$p(t) = \begin{cases} \frac{Ce^t}{1+Ce^t}, & p(t_0) < 1 \\ 1, & p(t_0) = 1 \\ \frac{Ce^t}{Ce^t-1}, & p(t_0) > 1, \\ 0, & p(t_0) = 0. \end{cases}$$

where C is determined by the initial value $p(t_0)$.

2. Homogeneous Equations

Definition. A function $f(x, y)$ is said to be *homogeneous of degree n* if for all suitable x, y and t ("suitable" in the sense that the arguments of f remains in the domain of definition),

$$f(tx, ty) = t^n f(x, y).$$

By the definition,

$$f(x, y) = xy + x^2, \quad g(x, y) = \frac{xy + y^2}{y^3 - x^2 y}, \quad h(x, y) = \frac{x+y}{x-y}$$

are homogeneous functions of degree $2, -1, 0$, respectively;

$$p(x, y) = xy + y^3 \quad q(x, y) = e^x y$$

are not homogeneous. (Note: You may remember the "homogeneous linear ODEs". Unfortunately, the same term "homogeneous" is used for a completely different meaning here.)

Definition. The differential equation

$$\frac{dy}{dx} = f(x, y)$$

is said to be *homogeneous* if $f(x, y)$ is a homogeneous function of degree 0.

Proposition. Any homogeneous first-order ODE becomes separable after the substitution

$$z = \frac{y}{x}.$$

Proof. Differentiating

$$y = zx$$

with respect to x gives

$$\frac{dy}{dx} = z + x\frac{dz}{dx}.$$

On the other hand, since $f(x, y)$ is homogeneous of degree zero,

$$f(x, y) = f(x^{-1}x, x^{-1}y) = f(1, z).$$

The original differential equation now translates to an equation in z:

$$z + x\frac{dz}{dx} = f(1, z).$$

This equation is clearly separable, and is equivalent to

$$\frac{dz}{f(1, z) - z} = \frac{dx}{x}.$$

Example. The equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$$

is by definition homogeneous. Hence, after the substitution $z = y/x$, translates to

$$\frac{dz}{z^{-1}} = \frac{dx}{x}.$$

Integrating on both sides gives the implicit solution

$$\frac{1}{2}z^2 = \ln|x| + C.$$

Translating back to y:

$$\frac{1}{2}y^2 = x^2 \ln|x| + Cx^2.$$