LECTURE 19: CONVOLUTION INTEGRALS

The definition of the *convolution* of two functions is straightforward.

Definition. Given $f(t), g(t)$ functions defined for $t \geq 0$, the *convolution* of f and g is defined as

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Properties.

- Commutative law: $f * g = g * f$.
  Proof. By a change of variable $s = t - \tau$, we have
  $$\begin{aligned}(f * g)(t) &= \int_0^t f(t-\tau)g(\tau)d\tau = -\int_t^0 f(s)g(t-s)ds \\ &= \int_0^t f(s)g(t-s)ds = (f * g)(t).\end{aligned}$$
- Distributive law: $f * (g_1 + g_2) = f * g_1 + f * g_2$.
  Proof. This follows immediately from the linearity of the integral.
- Associative law: $(f * g) * h = f * (g * h)$.
  Proof. By definition,
  $$\begin{aligned}((f * g) * h)(t) &= \int_0^t (f * g)(t-s)h(s)ds \\ &= \int_0^t \Big(\int_0^{t-s} f(t-s-\tau)g(\tau)d\tau\Big)h(s)ds \\ &= \int_0^t \int_0^{t-s} f(t-s-\tau)g(\tau)h(s)d\tau ds.\end{aligned}$$
  On the other hand,
  $$\begin{aligned}(f * (g * h))(t) &= \int_0^t f(t-s)(g * h)(s)ds \\ &= \int_0^t f(t-s)\int_0^s g(s-\tau)h(\tau)d\tau ds \\ &= \int_0^t \int_0^s f(t-s)g(s-\tau)h(\tau)d\tau ds \\ (\tilde{s} = \tau, \tilde{\tau} = s-\tau) &= \int_0^t \int_0^{t-\tilde{s}} f(t-\tilde{\tau}-\tilde{s})g(\tilde{\tau})h(\tilde{s})d\tilde{\tau}d\tilde{s}.\end{aligned}$$
  Clearly, $(f * g) * h = f * (g * h)$.

The following theorem tells us that the inverse Laplace of a product of functions is equal to the convolution of the respective inverse Laplace transforms of those functions.

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Theorem. If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$, where $F(s), G(s)$ are defined for $s > \alpha$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

Proof. It suffices to show that

$$\int_0^\infty (f * g)(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt \int_0^\infty g(t)e^{-st}dt.$$

The right hand side of this equality can be rewritten as

$$\int_0^\infty \int_0^\infty f(t)g(\tau)e^{-s(t+\tau)}d\tau dt.$$

Now, let $x = t + \tau$ be a substitute for τ with t fixed, the expression above is just

$$\begin{aligned}\int_0^\infty \int_t^\infty f(t)g(x-t)e^{-sx}dxdt &= \int_0^\infty \Big(\int_0^t f(t)g(x-t)dt\Big)e^{-sx}dx \\ &= \int_0^\infty (g * f)(t)e^{-sx}dx,\end{aligned}$$

where the first equality comes from reversing the order of integration in the region $\{t \geq 0, x \geq t\} \subset \mathbb{R}^2$. Finally, note that $g * f = f * g$. This completes the proof.

Example. In a previous lecture, it took us a fair amount of effort to find $\mathcal{L}^{-1}\{(1+s^2)^{-2}\}$. Now, by the theorem, we know immediately that

$$\begin{aligned}\mathcal{L}^{-1}\{(1+s^2)^{-2}\} = \mathcal{L}^{-1}\{(1+s^2)^{-1}\} * \mathcal{L}^{-1}\{(1+s^2)^{-1}\} &= \sin t * \sin t \\ &= \int_0^t \sin(t-\tau)\sin(\tau)d\tau \\ &= \int_0^t \frac{1}{2}[\cos(t-2\tau) - \cos t]d\tau \\ &= \frac{1}{4}[-\sin(t-2\tau) - 2\tau\cos t]\Big|_{\tau=0}^{t} \\ &= \frac{1}{2}(\sin t - t\cos t).\end{aligned}$$

Exercise. Consider $F(s)$ as $F(s)\cdot 1$ and note that $\mathcal{L}^{-1}\{1\} = \delta(t)$. Now apply the convolution theorem to finding $\mathcal{L}^{-1}\{F(s)\cdot 1\}$, what do you get?
Ans. $f(t) = \delta(t) * f(t)$. This means, the impulse function is playing the role of a "multiplicative identity" in the sense of convolution.

Example. Using convolution, solve the initial value problem

$$y'' + 2y' + 2y = g(t), \qquad y(0) = y'(0) = 0.$$

Applying $\mathcal{L}$ on both sides of this equation gives

$$(s^2 + 2s + 2)Y(s) = G(s).$$

Thus,

$$Y(s) = \frac{G(s)}{s^2 + 2s + 2}.$$

Therefore,

$$\begin{aligned} y(t) &= g(t) * \mathcal{L}^{-1}\{(s^2+2s+2)^{-1}\} = G(s) * \mathcal{L}^{-1}\{((s+1)^2+1)^{-1}\} \\ &= g(t) * (e^{-t}\sin t). \end{aligned}$$

Note that we obtained $e^{-t}\sin t$ only from the left hand side, i.e., the "system". And the output is simply the convolution of the input $g(t)$ and a function intrinsic to the system. Of course, this kind of result is neither restricted to this particular set of constant coefficients, nor to this particular set of initial values. As an exercise, you may try to convince yourself of this. And the result should generalize the remark at the end of section 3.