# LECTURE 16: INVERSE LAPLACE TRANSFORM, SOLVING INITIAL VALUE PROBLEMS

## 1. Inverse Laplace Transform

At the beginning of the current topic, we asked the question: Is the Laplace transform one-to-one? In other words, could two different functions share the same Laplace transform? Let's think. To begin with, we notice that, by definition, the Laplace transform is an integral, to be more precise, the limit of a sequence of definite integrals. What is a definite integral then? If we restrict ourselves to the notion of Riemann integrals (and there are broader notions of integrals out there), a definite integral is the limit of Riemann sums, as one divides the domain of integration into smaller pieces. For example, it is easy to show that, using this definition, the definite integral of $f(x) = 0$ over $[0, 2]$ is zero. Now, what if we change $f(x)$ a little bit and call it $g(x)$, say, $g(1) = 1$ and $g(x) = 0$ for $x \neq 1$. What is the definite integral of $g(x)$ over $[0, 2]$ then? Again, we try to find the Riemann sums for certain step size $\Delta x$, and we notice that the "step" that contains $x = 1$ would contribute a $1 \cdot \Delta x$ to the sum. However, this does not affect the integral in the end, since, as we take the limit $\Delta x \to 0$, the limit of the Riemann sum remains zero. What this example tells us is that if we change the value of a function at finitely many points, the Riemann integral of this function, if it exists, remains the same. In particular, if two "appropriate" functions differ only at finitely many points, then they share the same Laplace transform.

Now, if we define the *inverse Laplace transform* of a function $F(s)$ as any function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$ and ask, "Is the inverse Laplace transform unique?", then the reasoning in the previous paragraph would let us be more careful with what "unique" means. Certainly, as we have seen, the inverse Laplace transform is not unique as a function, since we could always change the value of $f(t)$ at finitely many points without changing $\mathcal{L}\{f(t)\}$. So it is reasonable to consider the uniqueness to be up to the difference at finitely many points. Formally, we have the following theorem.

**Theorem.**(Lerch) For a function $F(s)$, the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$, if it exists, is unique in the sense that we allow a difference of function values on *a set that has zero Lebesgue measure* (reads: a set that is negligible in integrals).

For example, since we have $\mathcal{L}\{1\} = s^{-1}$, by the theorem, we know that $\mathcal{L}^{-1}\{s^{-1}\} = 1$ and that the 1 on the right hand side of this equality is a *representative* of all the functions that has the Laplace transform being $s^{-1}$, which are equal to $f(t) = 1$ except on negligible sets.

In practice, how can one figure out the inverse Laplace transform of a function? Given what we know so far, one could look at the function and ask: (1) Is it the Laplace transform of some familiar function? (2) If not, does it appear like the Laplace transform of

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the derivative/shift/$t$-multiple of a familiar function? For example, consider the function

$$F(s) = \frac{1}{s^2 - 2s + 2}.$$

If we rewrite it as

$$F(s) = \frac{1}{(s-1)^2 + 1},$$

it is immediately recognizable that if we have a function $g(t)$ whose Laplace transform is

$$G(s) = \frac{1}{1 + s^2},$$

then $F(s)$ is $G(s)$ shifted to the right by 1, and by the first shift theorem, $f(t)$ is simply $e^t g(t)$. By our list, we have $g(t) = \sin t$, hence,

$$f(t) = e^t \sin t.$$

You may ask, why do we write the function in the form $((s-1)^2+1)^{-1}$ in the first place? Instead of forcing an answer like "we usually look for the patterns...", perhaps a more honest reply to this is: practice.

## 2. Solving Initial Value Problems

In this section, we turn to another question asked in the last lecture: How to apply the Laplace transform? Our current answer is, solving differential equations. The idea is simple: After a Laplace transform, certain *differential equations* of $y(t)$ get transformed into *algebraic equations* of $F(s)$ whose solutions are almost evident. Once we've found $F(s)$, apply the inverse Laplace transform to get $y(t)$. This is summarized in the diagram below:

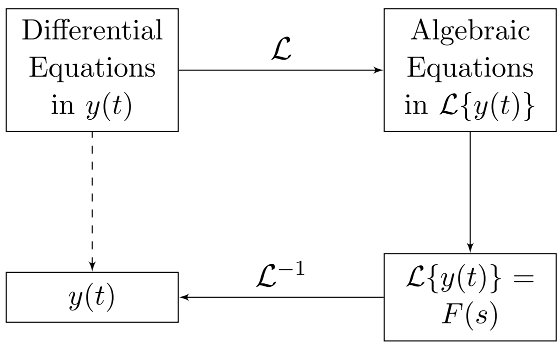


Figure 1. Solving Differential Equations Using Laplace Transform

**Example 1.** We apply the Laplace transform to solving the initial value problem:

$$y'' - y' - 2y = 0, \qquad y(0) = 1, y'(0) = 0.$$

First, transform on both sides of the equation and use the initial conditions, we have respectively:

$$\begin{aligned} \mathcal{L}\{y'' - y' - 2y\} &= (s^2F(s) - sy(0) - y'(0)) - (sF(s) - y(0)) - 2F(s) \\ &= (s^2 - s - 2)F(s) - s + 1, \\ \mathcal{L}\{0\} &= 0. \end{aligned}$$

Hence,

$$F(s) = \frac{s-1}{s^2-s-2} = \frac{s-1}{(s-2)(s+1)}.$$

Note that we are in a situation of using partial fractions, so let

$$\frac{A}{s-2} + \frac{B}{s+1} = \frac{s-1}{(s-2)(s+1)}.$$

This gives us

$$A(s+1) + B(s-2) = (A+B)s + (A-2B) = s-1.$$

Hence, comparing the coefficients,

$$A+B=1, \qquad A-2B=-1.$$

We get

$$A = \frac{1}{3}, \qquad B = \frac{2}{3}.$$

Thus,

$$F(s) = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}.$$

By the linearity of the Laplace transform, first shift theorem, and the fact that $\mathcal{L}\{1\} = \frac{1}{s}$, we obtain

$$y(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

as the solution to the initial value problem.

As we can see, this procedure is quite "algorithmic". However, it may sometimes require a bit of thought, and experience, to cope with the last step: finding the inverse Laplace transform of $F(s)$. Here is an example in which $\mathcal{L}^{-1}\{F(s)\}$ may not look obvious at first sight.

**Example 2.** We apply the Laplace transform to solving the initial value problem

$$y'' + y = \sin t, \qquad y(0) = 2, y'(0) = 1.$$

The first steps are routine:

$$\begin{aligned}\mathcal{L}\{y''+y\} &= s^2F(s) - y'(0) - sy(0) + F(s) = (s^2+1)F(s) - 2s - 1,\\ \mathcal{L}\{\sin t\} &= \frac{1}{s^2+1}.\end{aligned}$$

Thus,

$$F(s) = \frac{1}{(1+s^2)^2} + \frac{1+2s}{1+s^2}.$$

Now, in the expression of $F(s)$, we know pretty well what the inverse Laplace transform of the second term is, however, what is $\mathcal{L}^{-1}\{(1+s^2)^{-2}\}$? This is not obvious from the list of Laplace transforms, nor from the shifting properties. However, one may realize that this has something to do with the derivative of the functions $s(1+s^2)^{-1}$ and $(1+s^2)^{-1}$, whose inverse Laplace transforms are well known. Also, we might need to used the derivative property:

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s).$$

So let us try this out:

$$\begin{aligned}\frac{d}{ds}\frac{s}{1+s^2} &= \frac{1}{1+s^2} - \frac{2s^2}{(1+s^2)^2} = \frac{1-s^2}{(1+s^2)^2} \\ &= \frac{-1-s^2+2}{(1+s^2)^2} \\ &= \frac{2}{(1+s^2)^2} - \frac{1}{1+s^2}.\end{aligned}$$

Therefore,

$$\frac{1}{(1+s^2)^2} = \frac{1}{2}\frac{d}{ds}\frac{s}{1+s^2} + \frac{1}{2}\frac{1}{1+s^2}.$$

Hence, we can use the linearity and one of the derivative properties and get

$$\mathcal{L}^{-1}\{(1+s^2)^{-2}\} = -\frac{1}{2}t\cos t + \frac{1}{2}\sin t.$$

Conclusion,

$$\begin{aligned}y(t) &= \mathcal{L}^{-1}\{(1+s^2)^{-2} + (1+2s)(1+s^2)^{-1}\} \\ &= -\frac{1}{2}t\cos t + \frac{3}{2}\sin t + 2\cos t.\end{aligned}$$