LECTURE 14: REGULAR SINGULAR POINTS, EULER EQUATIONS

1. Regular Singular Points

During the past few lectures, we have been focusing on second order linear ODEs of the form $y'' + p(x)y' + q(x)y = g(x)$. Particularly, we learned that the series solutions of this type of equations are *valid* and unique on the common intervals where $p(x), q(x), g(x)$ are analytic. In this lecture, we attempt to solve equations near an a point at which one or more of p, q, g is not analytic.

Definition 1. (Ordinary/Singular Points) A point $x = x_0$ is an *ordinary point* of the second order linear ordinary differential equation

$$y'' + p(x)y' + q(x) = g(x)$$

if $p(x), q(x), g(x)$ are all *analytic* at the point x_0. Points that are not ordinary are called *singular points* of the differential equation.

Furthermore, the singularities of second order linear ODEs have been divided into two kinds, regular singularities and irregular singularities:

Definition 2. (Regular/Irregular Singularities) If $x = x_0$ is a singular point of the equation

$$y'' + p(x)y' + q(x) = g(x)$$

and $(x - x_0)p(x), (x - x_0)^2 q(x)$ are analytic at x_0, then x_0 is called a *regular* singularity. Singularities that are not regular are called *irregular* singularities.

As we can see, to test whether a point is ordinary or singular, one has to test whether it is a point at which certain functions are analytic. It turns out there are many ways for functions to be non-analytic at a point, to name a few:

(1) The function is not defined at $x = x_0$. Example: $f(x) = (x - x_0)^{-1}$.
(2) The function is defined but not continuous at $x = x_0$. Example:

$$f(x) = \begin{cases} 1, & x \geq x_0 \\ 0, & x < x_0 \end{cases}.$$

(3) The function is continuous but *not smooth* at $x = x_0$. Note: A function is said to be smooth at x_0 if all its derivatives at x_0 exist. Example:

$$f(x) = \begin{cases} (x - x_0)^2, & x \geq x_0 \\ -(x - x_0)^2, & x < x_0 \end{cases},$$

where the second derivative of $f(x)$ at $x = x_0$ does not exist.
(4) The function is smooth at $x = x_0$, but its Taylor expansion at x_0 does not converge

Date: 09/27/16.

to the function itself, i.e., the Taylor expansion at x_0 is not *valid*. Example:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

This function is smooth, but all its derivatives at $x_0 = 0$ are zero. In other words, the Taylor expansion of $f(x)$ at x_0 is zero. Clearly, this Taylor expansion is not valid.

Now we explain why there might be a problem of finding series solutions about a singular point. Consider the equation:

$$y'' + \frac{1}{x^2}y' - y = 0.$$

To solve it near $x_0 = 0$, we first multiply it by x^2 to make the series methods easier to apply:

$$x^2y'' + y' - x^2y = 0.$$

If one considers using the method of successive differentiation, solution shall be put in the form

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + ...,$$

but one notices immediately that it is impossible to calculate $y''(0)$ using the differential equation, since the coefficient of y'' is zero at $x = 0$.

What about the method of undetermined coefficients then? As a familiar set-up, let

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3...$$

and plug this into the equation, getting:

$$x^2(2a_2 + 6a_3x + ...) + (a_1 + 2a_2x + 3a_3x^2 + ...) - x^2(a_0 + a_1x + a_2x^2 + ...) = 0.$$

Grouping the powers of x setting the coefficients of x^i to vanish, we get:

$$\begin{aligned} a_1 &= 0, \\ 2a_2 &= 0, \\ 2a_2 - a_0 + 3a_3 &= 0, \\ &... \end{aligned}$$

Recall that $a_1 = y'(0)$, hence the above relations force $y'(0) = 0$. (This is in contrast with the case when the equation satisfies the existence/uniqueness condition, where $y'(0)$ can be freely specified as part of the initial condition.) In addition, we are not sure of the convergence of the this series, because $\frac{1}{x^2}$ is not analytic at $x = 0$, and the convergence theorem from the previous lecture does not apply.

It seems that one could say little about solutions near a singular point, but it turns out that there is a well-established method, called *method of Frobenius*, which gives valid (Frobenius) series solutions in a neighborhood of *regular* singularities. We are not going to introduce the Frobenius method in this course, due to the schedule. Instead, we are going to consider a special type of equations with a regular singularity at $x_0 = 0$, called the *Euler equations*. Hopefully, from this we can get a hint of why the *regular singularities* are "nice" singularities to have.

2. Euler Equations

Second order linear ODEs of the form

$$x^2y'' + \alpha xy' + \beta y = 0,$$

where α, β are real constants are called the *Euler Equations*. From our previous discussion, these equations have a regular singularity at $x_0 = 0$ as long as α, β are not both zero.

What about the solutions? Let us check whether there is any solution that is in the form

$$y(x) = x^r,$$

for some constant r.

Remark: In fact, this is not the first time that we attempt to solve an equation by asking "Are there solutions in the form...?" Think about the second order constant coefficient linear ODEs. There, we checked solutions in the form e^{rx}, and it worked!

Now, plugging $y = x^r$ in the equation, we obtain

$$x^2(x^r)'' + \alpha x(x^r)' + \beta x^r = 0,$$

that is,

$$r(r-1)x^r + \alpha r x^r + \beta x^r = (r^2 + (\alpha - 1)r + \beta)x^r = 0.$$

This reduces to

$$r^2 + (\alpha - 1)r + \beta = 0,$$

which is a quadratic equation in r. Let r_1, r_2 be the roots of $P(r) = r^2 + (\alpha - 1)r + \beta$. Depending on the values of α, β, there are three possibilities:
(1) r_1, r_2 are real and distinct;
(2) r_1, r_2 are repeated (real) roots;
(3) r_1, r_2 are complex conjugates of each other.

Case (1). We obtain two solutions

$$y_1(x) = x^{r_1}, \qquad y_2(x) = x^{r_2}$$

of the original equation. Since $r_1 \neq r_2$, it is easy to check the Wronskian and conclude that these two solutions are linearly independent. Hence the general solution of the equation is

$$y(x) = c_1x^{r_1} + c_2x^{r_2}.$$

However, you may notice a problem here: What if, for instance, $r_1 = \frac{1}{2}$? The solution is not defined for $x < 0$. For now, we confine to the domain $x > 0$ and leave the $x < 0$ case to the end of this section. Thus, the current result of the general solution should be

$$y(x) = c_1x^{r_1} + c_2x^{r_2}, x > 0.$$

Case (2). Here, you may recall a homework exercise we did for the second order homogeneous CCLDEs: to obtain a second solution in the case of repeated roots, we took the difference quotient

$$\frac{e^{r_1x} - e^{r_2x}}{r_1 - r_2},$$

and took the limit $r_1 \to r_2$. By using the L'hôpital's rule, we obtained the solution xe^{r_2x}. You may realize, what taking the difference quotient and the limit is doing is essentially taking the derivative of e^{rx} with respect to r and evaluate it at $r = r_2$. To work along this idea, let L denote the linear operator such that the equation can be written as $L[y] = 0$. Then $L[e^{rx}] = Q(r)e^{rx}$, where $Q(r)$ is the characteristic polynomial. Since $Q(r)$ has repeated roots $r_1 = r_2$, it must be in the form $c(r - r_2)^2$ for some nonzero constant c. Therefore,

$$\frac{\partial}{\partial r}L[e^{rx}] = \frac{\partial}{\partial r}(c(r-r_2)^2 e^{rx}) = 2c(r-r_2)e^{rx} + c(r-r_2)^2 xe^{rx}.$$

Clearly,

$$\frac{\partial}{\partial r}L[e^{rx}]\Big|_{r=r_2} = 0.$$

On the other hand,

$$\frac{\partial}{\partial r}L[e^{rx}] = L[\frac{\partial}{\partial r}e^{rx}].$$

Exercise. Show that if $L = D^2 + p(x)D + q(x)$, where $D = \frac{d}{dx}$, then

$$\Big(\frac{\partial}{\partial r}L\Big)f(x,r) = L\Big(\frac{\partial}{\partial r}f(x,r)\Big),$$

for any differentiable function $f(x, r)$.

Therefore,

$$0 = \frac{\partial}{\partial r}L[e^{rx}]\Big|_{r=r_2} = L[\frac{\partial}{\partial r}e^{rx}\Big|_{r=r_2}] = L[xe^{r_2 x}],$$

and $y(x) = xe^{r_2 x}$ is a solution of the equation $L[y] = 0$.

Now, back to the Euler equations, let

$$\tilde{L} = x^2 D^2 + \alpha x D + \beta.$$

Again, by the exercise above, we have

$$\Big(\frac{\partial}{\partial r}\tilde{L}\Big)x^r = \tilde{L}\Big(\frac{\partial}{\partial r}x^r\Big) = \tilde{L}[x^r \ln x].$$

On the other hand, by the assumption that $r_1 = r_2$,

$$\tilde{L}[x^r] = (r - r_1)(r - r_2)x^r = (r - r_1)^2 x^r.$$

Easily, one could check

$$\Big(\frac{\partial}{\partial r}\tilde{L}\Big)\Big|_{r=r_1} x^r = \frac{\partial}{\partial r}\Big|_{r=r_1}((r-r_1)^2 x^r) = 0.$$

Hence,

$$\tilde{L}[x^{r_1}\ln x] = 0,$$

and $y(x) = x^{r_1}\ln x$ is a solution.

Again, one could use the Wronskian to check that x^{r_1} and $x^{r_1}\ln x$ are linearly independent. Therefore, the general solution is

$$y(x) = (c_1 + c_2 \ln x)x^{r_1}, \qquad x > 0,$$

where we restrict to $x > 0$ because of $\ln x$.

Case (3). $P(r)$ has complex roots $a + bi$ and $a - bi$. Formally, the solutions are

$$y_1(x) = x^{a+bi}, \qquad y_2(x) = x^{a-bi}.$$

Two problems: One, how to make sense of x^r when r is a complex number? Two, $y_1(x)$ and $y_2(x)$ may be complex valued.

Definition: x^r is defined as $e^{r \ln x}$ for $r \in \mathbb{C}$.

Using this definition, we have

$$\begin{aligned} y_1(x) &= e^{(a+bi)\ln x} = e^{a \ln x}(\cos(b \ln x) + i \sin(b \ln x)), \\ y_2(x) &= e^{(a-bi)\ln x} = e^{a \ln x}(\cos(b \ln x) - i \sin(b \ln x)). \end{aligned}$$

Taking real and imaginary parts, we could obtain two real solutions:

$$\tilde{y}_1(x) = e^{a \ln x} \cos(b \ln x), \qquad \tilde{y}_2(x) = e^{a \ln x} \sin(b \ln x).$$

Again, one could check linear independence using the Wronskian. Thus, the general solution is

$$\begin{aligned} y(x) &= e^{a \ln x}(c_1 \cos(b \ln x) + c_2 \sin(b \ln x)) \\ &= x^a(c_1 \cos(b \ln x) + c_2 \sin(b \ln x)), \qquad x > 0. \end{aligned}$$

What about $x < 0$? Consider the change of variable $x = -t$. We have $\frac{d}{dt} y = -\frac{d}{dx} y$ and $\frac{d^2}{dt^2} y = \frac{d^2}{dx^2} y$. The Euler equation becomes

$$x^2 \frac{d^2}{dx^2} y + \alpha x \frac{d}{dx} y + \beta y = t^2 \frac{d^2}{dt^2} y + \alpha t \frac{d}{dt} y + \beta y = 0.$$

Hence, to solve for the Euler equation in $x < 0$, it suffices to solve

$$t^2 \frac{d^2}{dt^2} y + \alpha t \frac{d}{dt} y + \beta y = 0$$

for $t > 0$, which is also an Euler equation. Finally, substitute back $t = -x$ to obtain solutions for $x < 0$.

Therefore, we conclude that the general solution of an Euler equation about $x = 0$ is

$$y(x) = \begin{cases} c_1 |x|^{r_1} + c_2 |x|^{r_2}, & r_1 \neq r_2 \in \mathbb{R} \\ (c_1 + c_2 \ln |x|)|x|^{r_1}, & r_1 = r_2 \\ |x|^a (c_1 \cos(b \ln |x|) + c_2 \sin(b \ln |x|)), & r_1, r_2 = a \pm bi \end{cases}, \qquad x \neq 0.$$

3. Remark

In fact, there is a quicker way to obtain the solutions of the Euler equation for $x > 0$. If one makes the substitution

$$t = \ln x,$$

then the equation in $y(x)$

$$x^2 y'' + \alpha x y' + \beta y = 0$$

is turned in to the equation in $y(t)$

$$\ddot{y} + (\alpha - 1)\dot{y} + \beta y = 0.$$

Note that the latter is simply a constant coefficient homogeneous second-order ODE. One may easily find the solutions with the characteristic polynomials, then substitute back to having x as the independent variable.