LECTURE 13: SERIES SOLUTIONS NEAR AN ORDINARY POINT II

Last time, we have seen two series methods which are useful in finding solutions of linear second order ODEs. The ideas are assuming the solution to be analytic near the point $x = x_0$ and use both the equation and the initial value $(y(x_0), y'(x_0))$ to determine the coefficients in the Taylor expansion of the solution. In practice, it is sometimes possible to obtain a general formula of these coefficients, but more often, calculating by hand, one may only obtain the first few terms in the expansion. This may be regarded as a limitation of using the series method.

In this lecture, assuming that the computing power has no restriction, we discuss the more intrinsic "limitations" one has to take care of when using the series method. The question in mind is: Suppose that a (formal) series solution is found for a differential equation, what is its radius of convergence?

1. More About Power series

Here we list more facts about the power series, which will be quite useful for finding the (minimal) interval of convergence of certain functions.

Fact 1. Let $\sum_{i=0}^{\infty} a_i(x-x_0)^i$ and $\sum_{i=0}^{\infty} b_i(x-x_0)^i$ be two power series converging to $f(x), g(x)$ on $|x-x_0| < r$, respectively. Then their sum and difference, defined as $\sum_{i=0}^{\infty}(a_i \pm b_i)(x-x_0)^i$ is a power series converging to $f(x) \pm g(x)$ on $|x-x_0| < r$.

Fact 2. Let $\sum_{i=0}^{\infty} a_i(x-x_0)^i$ and $\sum_{i=0}^{\infty} b_i(x-x_0)^i$ be two power series converging to $f(x), g(x)$ on $|x-x_0| < r$, respectively. Then their product, defined as $\sum_{i=0}^{\infty} c_i x^i$ where $c_i = \sum_{k=0}^{i} a_k b_{i-k}$, is a power series converging to $f(x)g(x)$ on $|x-x_0| < r$.

Fact 3. Let $P(x)$ be a polynomial. Given $x_0 \in \mathbb{R}$, let z_0 be the complex root of $P(x)$ which is closest to x_0 in the complex plane. Then the Taylor expansion of $(P(x))^{-1}$ converges to $(P(x))^{-1}$ at least on $|x-x_0| < \|z_0 - x_0\|$, where $\|\cdot\|$ denotes the Euclidean norm on the complex plane:

$$\|a+bi\| = (a^2+b^2)^{\frac{1}{2}}.$$

Examples.

(1) By explicit calculation, we see that the Taylor expansions at $x = 0$ of $f(x) = (1+x)^{-1}$ and $g(x) = (x-1)^{-1}$ are convergent on $|x| < 1$. Therefore, the Taylor expansion of $f(x)g(x) = (1-x^2)^{-1}$ is convergent at least on $|x| < 1$.

(2) Since the Taylor expansion of $\sin x$ has the interval of convergence $(-\infty, \infty)$, by Fact 2 and the result in (1), we know that the Taylor expansion of $\sin x(1-x^2)^{-1}$ at $x = 0$ is convergent at least on $|x| < 1$.

(3) Note that $1 + x^2$ has two complex roots $\pm i$. For $x_0 = 1$, by Fact 3, we have $\|z_0 - x_0\| = \sqrt{2}$. Therefore, the Taylor expansion of $(1+x^2)^{-1}$ is convergent to the

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function itself at least on the interval $|x-1|<\sqrt{2}$.

2. Equations of the Form $y''+p(x)y'+q(x)y=g(x)$

Definition. A function $f(x)$ is said to be *analytic* at x_0 if (a) it has a Taylor expansion in powers of $(x-x_0)$ on $|x-x_0|<r$ for some $r>0$, and (b) the Taylor expansion is *valid*, i.e., converges to the values of $f(x)$, on the interval $|x-x_0|<r$.

Theorem. In the equation

$$y''+p(x)y'+q(x)y=g(x),$$

if each function $p(x), q(x), g(x)$ are analytic at $x=x_0$ and their Taylor expansions are valid on $|x-x_0|<r$, then there is a *unique* solution $y(x)$ of the equation satisfying the initial conditions

$$y(x_0)=a_0, \qquad y'(x_0)=a_1.$$

Moreover, this solution is analytic at x_0 and its Taylor expansion is *valid* on $|x-x_0|<r$.

Example 1. Recall the three equations which we studied in the last lecture,

$$y''+y=0, \qquad y''-xy'+x^2y=0, \qquad y''+\cos xy'+e^xy=0.$$

In each of these equations, the coefficient functions are analytic and the Taylor expansions at any x_0 are valid on $(-\infty,\infty)$. Therefore, the power series solutions we obtained are valid for all x.

Example 2. Consider the initial value problem

$$y''+\frac{x}{1+x^2}y'-\frac{1}{1+x^2}y=0, \qquad y(0)=1, y'(0)=1,$$

which is equivalent to

$$(1+x^2)y''+xy'-y=0, \qquad y(0)=1, y'(0)=1.$$

The second form is more convenient for us to apply the series method, while the first form is used to figure out the interval on which the series solution is valid.

By Fact 3, we know that the Taylor expansion at $x=0$ of $(1+x^2)^{-1}$ is valid on $|x|<1$. Of course, the Taylor expansion of x at $x=0$ is valid for all x. Hence, by Fact 2, $x(1+x^2)^{-1}$ has the interval of convergence being at least $|x|<1$. Therefore, the series solution we could find is valid at least on the interval $(-1,1)$.