LECTURE 12: SERIES SOLUTIONS NEAR AN ORDINARY POINT: I (CONT.)

1. Initial Value at $x_0 \neq 0$; Example

Last time we applied the series method to a couple of second order ODEs for which the initial value is set at $x_0 = 0$, hence the power series solution is at $x = 0$. Of course, one may ask, what if the initial value is given at some $x_0 \neq 0$? The answer is simple, use the "shifting trick" to let $t = x - x_0$. By the chain rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}, \quad \text{etc.}$$

Hence, the initial value problem can be translated into one of $y(t)$ with initial value being set at $t_0 = 0$. This shall be clear through an example.

Example:(Initial Value at $x = 1$) Consider the initial value problem

$$y'' - xy' + x^2 y = 0, \qquad y(1) = 1, \quad y'(1) = 1.$$

Note that the initial value is set at $x_0 = 1$, hence we make the substitution $t = x - 1$. Then the initial value problem is turned into one in $y(t)$:

$$y'' - (t+1)y' + (t+1)^2 y = 0, \qquad y(0) = 1, \quad y'(0) = 1.$$

(Note that, now, all the derivatives are taken with respect to t.) Let's assume that a power series solution take the form

$$y(t) = \sum_{n=0}^{\infty} a_n t^n,$$

and note that, by the initial values, we have

$$a_0 = 1, \quad a_1 = 1.$$

Thus,

$$y' = \sum_{n=1}^{\infty} n a_n t^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}.$$

Plugging $y(t)$ in the equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} - (t+1)\sum_{n=1}^{\infty} n a_n t^{n-1} + (t+1)^2 \sum_{n=0}^{\infty} a_n t^n = 0.$$

Now you may test your proficiency in shifting the summation indices by turning the left hand side into the form:

$$\{\ \textit{a few isolated terms with smaller indices}\} + \textstyle\sum_{n=k}^{\infty} [\textit{coefficient formula}] \cdot x^n.$$

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In fact, the first term on the LHS of the equality above equals to:

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} = \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m;$$

the second term:

$$-(t+1)\sum_{n=1}^{\infty} n a_n t^{n-1} = -\sum_{m=1}^{\infty} m a_m t^m - \sum_{m=0}^{\infty}(m+1)a_{m+1}t^m;$$

and the third term:

$$\begin{aligned}(t+1)^2\sum_{n=0}^{\infty} a_n t^n &= (t^2+2t+1)\sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{m=2}^{\infty} a_{m-2}t^m + \sum_{m=1}^{\infty} 2a_{m-1}t^m + \sum_{m=0}^{\infty} a_m t^m.\end{aligned}$$

Putting together, we have

$$\left(\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m\right) + \left(-\sum_{m=1}^{\infty} m a_m t^m - \sum_{m=0}^{\infty}(m+1)a_{m+1}t^m\right) + \left(\sum_{m=2}^{\infty} a_{m-2}t^m + \sum_{m=1}^{\infty} 2a_{m-1}t^m + \sum_{m=0}^{\infty} a_m t^m\right) = 0,$$

which is

$$(2a_2 - a_1 + a_0) + (6a_3 - 2a_2 + 2a_0)t + \sum_{m=2}^{\infty}[(m+2)(m+1)a_{m+2} - (m+1)a_{m+1} - (m-1)a_m + 2a_{m-1} + a_{m-2}]t^m.$$

Therefore, the recurrence relations for the coefficients are

$$\begin{aligned} a_0 &= 1, \\ a_1 &= 1, \\ 2a_2 - a_1 + a_0 &= 0, \\ 6a_3 - 2a_2 + 2a_0 &= 0, \\ (m+2)(m+1)a_{m+2} - (m+1)a_{m+1} - (m-1)a_m + 2a_{m-1} + a_{m-2} &= 0, \quad m \geq 2. \end{aligned}$$

Of course, to obtain a solution of the original equation, substitute back using

$$t = x - 1.$$

2. $y'' + p(x)y' + q(x)y = 0$, $p(x), q(x)$ NOT POLYNOMIALS.

The examples which we have considered until now are those with $p(x), q(x)$ being polynomials. As you can imagine, these are the cases in which one could certainly obtain a recurrence relation when applying the method of undetermined coefficients (for series solutions). However, there are examples in which a general recurrence formula for the coefficients is not as easy to be found, but one still could find the first any number of coefficients in a series solution. See the following example.

Example: Consider the equation

$$y'' + \cos x y' + e^x y = 0.$$

Assuming that solution near $x = 0$ can be expanded as a power series:

$$y(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

then,

$$y'(x) = (a_0 + a_1 x + a_2 x^2 + ...)' = a_1 + 2a_2 x + 3a_3 x^2 + ... = \sum_{k=1}^{\infty} k a_k x^{k-1},$$

$$y''(x) = (a_1 + 2a_2 x + 3a_3 x^2 + ...)' = 2a_2 + 3 \cdot 2a_3 x + ... = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

Also note that

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - ...$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

Plugging these expansions in the equation, and rearranging the terms according to the powers of x, we obtain

$$\begin{aligned} 0 = &(2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + ...) \\ &+ (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + ...)(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + ...) \\ &+ (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 ...) \\ = &(2a_2 + a_1 + a_0) + (6a_3 + 2a_2 + a_1 + a_0)x + (12a_4 + \tfrac{1}{2}a_1 + 3a_3 + a_2 + \tfrac{1}{2}a_0)x^2 \\ &+ (20a_5 + 4a_4 + a_3 + \tfrac{1}{2}a_1 + \tfrac{1}{6}a_0)x^3 + ... \end{aligned}$$

Therefore,

$$\begin{aligned} a_2 &= -\frac{a_1 + a_0}{2}, \\ a_3 &= -\frac{2a_2 + a_1 + a_0}{6} = 0, \\ a_4 &= -\frac{a_1 + 2a_2 + a_0}{24} = 0, \\ a_5 &= -\frac{3a_1 + a_0}{120}, \\ &...... \end{aligned}$$

and

$$y(x) = a_0(1 - \frac{1}{2}x^2 - \frac{1}{120}x^5 + ...) + a_1(x - \frac{1}{2}x^2 - \frac{1}{40}x^5 + ...).$$

Here, we've specified the terms up to order five in a general solution of the given equation.