LECTURE 1: INTRODUCTION, LINEAR EQUATIONS, INTEGRATING FACTORS

1. Introduction

Differential equations are equations of an unknown function which involve taking derivatives. If the unknown function depends on one independent variable, the equation is called an *ordinary differential equation* (ODE); if it depends on more than one independent variable, the equation is called a *partial differential equation* (PDE).

You may have learned some differential equations before. For example, newton's second law of motion, $F = ma$, may be interpreted as an ODE. A particular case is the *free-fall equation* (without air resistance):

$$\frac{d^2s}{dt^2} = g \approx 9.8m/s^2,$$

where s is the vertical distance relative to some fixed height. Note that here the derivative is taken with respect to the time variable t. Often, when this is the case, each $\frac{d}{dt}$ may be replaced by an extra dot on top of the unknown function, say,

$$\frac{ds}{dt} \equiv \dot{s}, \quad \frac{d^3u}{dt^3} \equiv \dddot{u}, \quad \text{etc.}$$

However, if the derivatives taken are more than three, the "dot" notation is rarely used for practical (aesthetic) reasons.

One could consider another version of free-fall equation, but with air resistance, which is assumed to be proportional to the velocity $\frac{ds}{dt}$:

$$\frac{d^2s}{dt^2} = g - \lambda\frac{ds}{dt}.$$

You may also recall the *population model* (without environmental constraints) from your calculus course:

$$\frac{dp}{dt} = rp,$$

where p is the population function and r is a constant understood as the reproduction rate. If one assumes in addition that there is a population cap that the environment allows, say, C, then it is easy to see that the following equation has taken this extra information into account:

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{C}\right).$$

If p is close to zero, the rate of growth $\frac{dp}{dt}$ is close to rp, i.e., p grows exponentially. However, when p is close to C, the right hand side approaches zero and the growth rate slows down. If p is greater than C, the right hand side of the equation is negative, so the population must decrease. To this point we see that qualitatively this equation fits our expectation quite well.

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Question. Is the following equation an acceptable model for population growth with environmental cap C? Explain why.

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{C}\right)^2.$$

As for PDEs, here are some standard examples:

Heat Transfer in a Rod:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

where u stands for the temperature, x is the variable in the space dimension, t the time variable. Sometimes, partial derivatives such as $\frac{\partial}{\partial x}$ are replaced by the name of the independent variable appearing at the right-bottom of the unknown function, for example,

$$\frac{\partial^2 u}{\partial x^2} \equiv u_{xx}, \quad \frac{\partial^3 u}{\partial x^2 \partial y} \equiv u_{xxy}, \quad \text{etc.}$$

Wave Equation in $\mathbb{R}^2$:

$$a^2(u_{xx} + u_{yy}) = u_{tt}.$$

Laplace Equation in $\mathbb{R}^3$:

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

In general, in $\mathbb{R}^n$ with coordinates $(x_1, ..., x_n)$, define the Laplace operator Δ to be

$$\Delta := \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2},$$

the heat, wave, and Laplace equation with *space dimension* n then have the short-hand expressions:

$$\Delta u = u_t, \quad a^2 \Delta u = u_{tt}, \quad \Delta u = 0,$$

respectively.

A less elementary example: The KdV Equation:

$$u_t + u_{xxx} + 6uu_x = 0.$$

This equation models waves on shallow water surfaces.

1.1. **The order of an equation.** The *order* of a differential equation is defined to be the maximum number of derivatives taken upon the unknown function.

By this definition, among the equations listed above, the population model equations are of *first* order; the free-fall equations, the heat, wave, Laplace equations are of *second* order; the *KdV* equation is of *third* order.

1.2. **Linearity of differential equations.** Note that any ODE of order n with independent variable x can be written as

$$f(x, y, y', ..., y^{(n)}) = 0,$$

where f is a function of $x, y, y', ..., y^{(n)}$. If, furthermore, f is expressible as a linear combination of $1, y, y', ..., y^{(n)}$ with coefficients being functions of x only, explicitly,

$$f(x, y, y', ..., y^{(n)}) = q_n(x)y^{(n)} + ... + q_1(x)y' + q_0(x)y + g(x),$$

then the equation is said to be *linear*.

Again, referring to the equations listed above, the free-fall equations and the population model without environmental capacity are linear. The population model with environmental capacity is nonlinear.

The notion of linearity for PDEs is similarly defined. Roughly speaking, a PDE

$$f(x, y, ..., u, u_x, u_y, ..., u_{xx}, u_{xy}, u_{yy}, ..., u_{xxx}, ...) = 0$$

($x, y...$ independent variables, u unknown function) is said to be linear if f is a linear function of u and all its partial derivatives. In other words, f is a linear combination of $1, u, u_x, u_y, ..., u_{xx}, ...$ whose coefficients are functions of the independent variables only.

Among the PDEs listed above, the heat, wave, Laplace equations are linear. The KdV equation is nonlinear because of the term $6uu_x$.

2. First Order Linear ODEs; Integrating Factors

2.1. **First order linear ODEs.** The definition of "order" and "linearity" in the previous section tells us that the first order ODEs must take the form

$$f(x, y, y') = s(x)\frac{dy}{dx} + q(x)y + r(x) = 0.$$

Assumed to be first order, the coefficient of y' in $f(x, y, y')$ needs to be nonzero. Thus, the equation can be further simplified to the form

$$\frac{dy}{dx} + p(x)y = g(x),$$

which we call *the standard form.*

The good news is that this kind of equations are always solvable by hand (modulo possible difficulty in taking anti-derivatives). One only needs to notice that

$$\frac{d}{dx}(e^{\int p(x)dx}y) = e^{\int p(x)dx}\left(\frac{dy}{dx} + p(x)y\right).$$

Multiplying the equation above by $\mu(x) = e^{\int p(x)dx}$ then gives:

$$\frac{d}{dx}(e^{\int p(x)dx}y) = e^{\int p(x)dx}g(x).$$

Therefore, the solution y can be found by direct integration, then divide by $e^{\int p(x)dx}$:

$$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}g(x)dx + C\right).$$

The multiplier $\mu(x) = e^{\int p(x)dx}$, is called an *integrating factor.* Its use makes the first order linear ODEs solvable direct integration. Here I give two examples:

Example 1. The linear equation

$$\frac{dy}{dx} + \frac{1}{x}y = 6x,$$

already in the standard form, has integrating factor

$$\mu(x) = e^{\int \frac{1}{x}dx} = x.$$

Multiplying both sides of the equation by $\mu(x)$ gives

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y = 6x^2.$$

Therefore,

$$y = \frac{1}{x}\left(\int 6x^2 dx + C\right) = 2x^2 + Cx^{-1}.$$

Example 2. The equation $(1+x^2)dy+2xydx = \cot x dx$ can be turned into the standard form

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cot x}{1+x^2}.$$

Thus, the integrating factor is

$$\mu(x) = e^{\int \frac{2x}{1+x^2}dx} = 1 + x^2.$$

And the solution

$$y = (1+x^2)^{-1}\int \cot x dx = (1+x^2)^{-1}(\ln|\sin x| + C).$$

However, if you are careful enough, the integrating factor is used rather "blindly" here. The left hand side of the original equation is already in the *exact* form $d((1+x^2)y)$, so there is really no need to first turn it into the standard form.