FINAL CHECKLIST

1. Topic Outline

Listed below is an outline of topics that we have covered in this course. Numerical methods and the derivation of heat/wave/Laplace/Sturm-Liouville equations are not in the list, even though they are covered as well.

1.1. First Order ODEs.

1.**Linear Equations**: Integrating factors. Idea: For the equation $y' + p(x)y + q(x) = 0$, find a function $\mu(x)$, such that whenever $y(x)$ is a solution to the equation, $z(x) = \mu(x)y(x)$ solves an equation of the form $z' + \tilde{q}(x) = 0$.

2.**Separable Equations**: Equations which can be put in the form $p(x) + q(y)\frac{dy}{dx} = 0$. Often, we obtain implicit solutions for separable equations.

3.**Solving non-separable equations via separable ones**: e.g. consider the equation $\frac{dy}{dx} = \frac{x^2+xy+y^2}{xy}$, and use the substitution $z = \frac{y}{x}$.

4.**Autonomous Equations**: Vector field plot and phase-line analysis. Equilibrium and stability.

5.**Existence & Uniqueness Theorem**: Statement of the Theorem. Note: theorem is about initial value problems; the conclusion is local, nearby our initial value.

6.**Modeling**: Keys are, identify the unknown function, and use the relations that can be derived from the statement of the problem to establish equations.

7.**Exact Equations**: Testing exactness. If exact, how to find solutions? If not exact, is there any test that can tell us whether we can multiply the entire equation by some function and make the equation exact?

1.2. Second Order Linear ODEs.

1.**Constant Coeff., Homogeneous**: Characteristic Polynomial.

2.**Constant Coeff., Non-homogeneous**: Undetermined Coefficients; or Variation of Parameters.

3.**Non-constant Coeff., Homogeneous**: One non-zero solution known, find another solution (linearly independent from the one already known) by Reduction of Order.

4.**Non-constant Coeff., Non-homogeneous**: From knowing a fundamental set of solutions of the underlying homogeneous equation to a particular solution of the non-homogeneous equation—*Variation of Parameters.*

1.3. Series Solutions of ODEs (at Ordinary Points).

1.**Power Series**: Definition of Power series about the point $x = x_0$; Radius of Convergence; Ratio test; Taylor expansion; Analytic Functions.

2.**Method of Undetermined Coefficients**: Expand the solution $y(x)$ about the point $x = x_0$ where the initial conditions are set, then use the equation to determine the coefficients in the expansion.

3.**Shift of Index and Recurrence Relation**: The summation notation (and shift of index) is useful only when we are interested in knowing more than a few terms in the series solution.

1.4. **Euler Equations.** Identify the form of an Euler equation; Three cases of solutions.

1.5. **Laplace Transform.**

1.**Definition and Calculation**: The Laplace transform; Laplace transform of elementary functions; First and second shifting theorems; Derivative properties; Inverse Laplace transform.

2.**Step Functions and Impulse Functions**: Definitions of $u_c(t)$ and $\delta(t)$; their Laplace transform.

3.**Convolution Integral**: Definition; Crucial theorem: $\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$.

4.**Laplace Transform and ODEs**: We only considered the case when the ODE has constant coefficients, but the forcing function can be various kinds.

1.6. **Two-point Boundary Values problems.**

The concept of *eigenvalues* and *eigenfunctions*.
Solving Principle: Find the general solutions of the equation; then use the boundary values to determine whether the solutions can be nonzero.

1.7. **Fourier Series.**

1.**Definition and Calculation**: Periodic functions; Definition of Fourier series; Formulas for calculating the Fourier coefficients.

2.**Fourier Convergence Theorem**: Statement of the theorem; Given a periodic function (maybe discontinuous), which function does its Fourier series converge to?

3.**Sine and Cosine Series**: Even and odd functions and properties (especially the integral over the interval $[-L, L]$); Expanding $f(x)$ $(0 \leq x < L)$ as a cosine or sine series of period $2L$.

1.8. **Heat, Wave, Laplace Equations.**

1.**Definitions**: Heat equation (with ends with fixed constant temperature or insulated ends); The meaning of “steady state”; Wave equation (with zero initial velocity or position, but you need to know how to bring them together); Laplace equation (in a rectangle or a round disk);

2.**Solution Technique**: The principle of solutions for these equations is three steps: Separation of variables, obtaining two ODEs; Consider the homogeneous boundary/initial values, giving us a two-point boundary value problem for one of the ODEs; use the remaining boundary/initial conditions, together with the principle of superposition to find the formal solution.

1.9. **Sturm-Liouville BVP.**

1.**Definitions**: The form of a Sturm-Liouville problem (including the linear operator L, which turns (SL) into the form $L[y] = \lambda r(x)y$); Hermitian L^2-inner product $\langle u, v\rangle_H$; L^2-inner product $\langle u, v\rangle_{r(x)}$; orthogonality of functions under $\langle \cdot, \cdot\rangle_{r(x)}$; normalized eigenfunctions;

2.**Theorems**: Lagrange identity (i.e., $\langle L[u], v\rangle_H = \langle u, L[v]\rangle_H$, where u, v satisfy the BV of (SL)) and its proof; Using the Lagrange identity to show all eigenvalues of (SL) are real, and that eigenfunctions corresponding to different eigenvalues are orthogonal under $\langle \cdot, \cdot\rangle_{r(x)}$; The statement and meaning of Theorems 4-5.

1.10. **PDEs with nonzero forcing terms.**

Two methods are useful for solving PDEs with nonzero forcing terms.
One, find a *steady-state solution* of the equation which also satisfies the boundary conditions; then use this steady-state solution to transform the IVBP into one that has homogeneous equation and boundary values.
Two, assuming that the boundary conditions of the IBVP are homogeneous; we study the associated two point BVP. If it is of Sturm-Liouville type with eigenfunctions ϕ_k, then one can write solution of the original IBVP as a combination $\sum b_k(t)\phi_k(x)$ (with time-varying coefficients). The homogeneous boundary values will then be automatically satisfied. The PDE will give rise to ODEs in $b_k(t)$. Initial values for these ODEs are contributed by the initial values of the IBVP.

---

2. Sample Review Questions

Here are some sample review questions for your reference, which are barely organized and not complete.

1. Give examples of a first order ODE that is (i) linear but not separable; (ii) separable but not linear; (iii) exact but not separable; (iv) exact but not linear? How would you solve the ODEs that you've written down? How to incorporate initial values in your solutions?

2. Give an example of a first order ODE which becomes exact after being multiplied by an integrating factor.

3. What is an autonomous ODE? Write down an autonomous equation, then use the phase line analysis to find its equilibria and the corresponding stability.

4. What are the methods of *reduction of order* and *variation of parameters* useful for?

5. What are the shifting properties of the Laplace transform? Is there any caution, to your experience, when applying these properties?

6. How to carry out the idea of using Laplace transform to solve *constant coefficient* ODEs?

7. How do you understand the impulse function? How does it relate to other functions, such as the step function?

8. Write down an ODE with initial values, then apply the series method (at an ordinary point) to solve it.

9. What does an Euler equation look like? How to find its solutions?

10. What is the principle of superposition? Under what conditions is this principle useful? What is its role in solving IBVP for PDEs (for example, the wave IBVP)?

11. Given a function $f(x)$, periodic with period $2L$, and f, f' both piecewise continuous, which is the limit of its Fourier series expansion?

12. Can we say that the Theorem 5 (convergence theorem of the ψ_n-series, where ψ_n are all the eigenfunctions of (SL)) for the Sturm-Liouville problem is a generalization of the Fourier convergence theorem? Why?

13. Briefly explain why in the heat equation in a rod, the boundary condition $u(0,t) = u(L,t) = 0$ will give us $X_n(x)$ as sine functions, while $u_x(0,t) = u_x(L,t) = 0$ will give us $X_n(x)$ as cosine functions.

14. Briefly explain why in the wave equation, the zero initial velocity gives us $T_n(t)$ as cosine functions, while the zero initial position gives us $T_n(t)$ as sine functions.

15. What does it mean for a differential operator, defined with certain boundary conditions to be satisfied, is self-adjoint? Why is self-adjointness desirable (in terms of knowing the behavior of eigenvalues, eigenfunctions)?

16. How to solve an IBVP with a nonzero forcing term, and (perhaps) nonzero constant boundary conditions? Practice.