HOMEWORK 4

Due: 09/26/2016

1. Textbook Exercises

5.1: 5, 13, 18, 21, 25;

5.2: 2, 10, 15;

Comments:
For the questions on series solutions which ask for first few coefficients in the series rather than a recurrence formula, both *successive differentiation* and *undetermined coefficients* are applicable. With practice, you may develop a taste as to when to use which.

2. Additional Questions

A1: Solve Question 42 in Prof. Nolen's *Additional Homework Problems.*

A2: Solve Question 49 in Prof. Nolen's *Additional Homework Problems.*

A3: Here I present a proof of the statement: *If a power series centered at zero, say,* $a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$, *is convergent at* $x = 1$, *then it must be absolutely convergent at all* x *satisfying* $|x| < 1$. (In particular, this implies that the set of x at which the power series converges absolutely must be an interval.) Your task is to fill in the marked "gap" within my argument below.

Proof. By the assumption, the series

$$a_0 + a_1 + ... + a_n + ...$$

converges. Let L be its limit. Thus, by the definition of convergence, for any $\epsilon > 0$, there exists a positive integer $N > 0$ such that for all $n \geq N$ the inequality

$$|a_0 + a_1 + ... + a_n - L| < \epsilon$$

holds. In particular, this implies that

$$\begin{aligned} |a_{n+1}| &= |(a_0 + ... + a_{n+1} - L) - (a_0 + ... + a_n - L)| \\ &\leq |a_0 + ... + a_{n+1} - L| + |a_0 + ... + a_n - L| \\ &< 2\epsilon \end{aligned}$$

for all $n \geq N$.

Now let us examine the series $|a_0| + |a_1x| + ... + |a_Nx^N| + |a_{N+1}x^{N+1}| + ...$, where $|x| < 1$. The sum of the first N terms is clearly finite. To see that the sum of the entire series is finite, it suffices to show that

$$|a_{N+1}x^{N+1}| + |a_{N+2}x^{N+2}| + ... < \infty,$$

when $|x| < 1$. Indeed, since $|a_{n+1}| < 2\epsilon$ for all $n \geq N$, ...

(GAP!)

Therefore, the original series converges absolutely at all $-1 < x < 1$.