EXAM I SOLUTIONS, MATH 353 FALL 2016

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. This test has **7 pages** with **5 questions and a bonus**. You have **50 minutes** to answer all the questions. *Good Luck* !

Useful Formula:

$$y_p(x) = -y_1(x) \int \frac{y_2(x)r(x)}{W(y_1, y_2)} dx + y_2(x) \int \frac{y_1(x)r(x)}{W(y_1, y_2)} dx,$$

or

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ r(x) \end{pmatrix}.$$

You are responsible for identifying correctly the situation in which these formulas can be applied.

Question	Max. Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total	100	
Bonus	10	

1. (20 points) For each of the following first-order equations, determine whether they are (i) *linear*; (ii) *separable*; (iii) *exact*; (iv) *homogeneous*. Briefly explain your reasoning.

(1) $$y^2 + y - x\frac{dy}{dx} = 0.$$

- **non-linear**: the y^2 term;
- **separable**: can be put in the form $\dfrac{y'}{y^2+y} = x^{-1}$;
- **non-exact**: $\dfrac{\partial}{\partial y}(y^2+y) = 2y+1 \neq \dfrac{\partial}{\partial x}(-x) = -1$;
- **non-homogeneous**: write as $y' = \dfrac{y^2+y}{x}$ and the right hand side is not a homogeneous function.

(2) $$(x^2 - 2y^2) + xy\frac{dy}{dx} = 0.$$

- **non-linear**: the $-2y^2$ or the xyy' term;
- **non-separable**: $y' = f(x,y) = \dfrac{2y^2 - x^2}{xy}$ and $f(x,y)$ cannot be written as p(x)q(y) for some functions p, q;
- **non-exact**: $M_y = -4y \neq N_x = y$;
- **homogeneous**: write as $y' = \dfrac{2y^2 - x^2}{xy}$ and the right hand side is a homogeneous function of zero degree.

(3) $$(2x\cos x - x^2\sin x)y + x^2\cos x\frac{dy}{dx} = 0.$$

- **linear**: The equation can be put in the form $y' + p(x)y = 0$;
- **separable**: Equations of the form $y' + p(x)y = 0$ are separable;
- **exact**: $M_y = 2x\cos x - x^2\sin x = N_x$;
- **non-homogeneous**: write as $y' = \dfrac{yx^2\cos x}{x^2\sin x - 2x\cos x}$ and the right hand side is not a homogeneous function of any degree.

2. (20 points) Solve the following first-order equations.

(1) $$x\frac{dy}{dx} - 3y = x^4.$$

Note that this is a linear equation which has the standard form

$$\frac{dy}{dx} - \frac{3}{x}y = x^3.$$

An integrating factor:

$$\mu(x) = e^{\int -\frac{3}{x}dx} = x^{-3}(\text{choice}).$$

Multiplying the original equation by x^{-3} yields

$$(x^{-3}y)' = x^{-3} \cdot x^3 = 1.$$

Thus

$$x^{-3}y = x + C,$$
$$y = x^4 + Cx^3.$$

(2) $$\left(x + \frac{2}{y}\right)\frac{dy}{dx} + y = 0.$$

This equation is exact. Check:

$$M(x, y) = y, \quad N(x, y) = x + \frac{2}{y} \Rightarrow M_y = 1 = N_x.$$

Hence,

$$F(x, y) = \int y dx + h(y) = xy + h(y).$$

$$F_y = x + h'(y) = x + \frac{2}{y}.$$

We have a choice of

$$h(y) = 2\ln|y|,$$

and the solutions of the differential equation is given implicitly by

$$xy + 2\ln|y| = C,$$

where C is a constant.

3. (20 points) Given that $y_1(x) = x$ is a solution of the second-order differential equation

$$x^2y'' - 2xy' + 2y = 0, \qquad x > 0,$$

first use *reduction of order* to find *all* homogeneous solutions, then use *variation of parameters* to find a particular solution of the non-homogeneous equation

$$x^2y'' - 2xy' + 2y = 2x^3, \qquad x > 0.$$

Reduction of Order: Assuming a second solution to be $y_2(x) = v(x) \cdot x$, we have

$$\begin{aligned} 0 =& x^2(vx)'' - 2x(vx)' + 2vx \\ =& x^2(v''x + 2v') - 2x(v'x + v) + 2vx \\ =& x^3v''. \end{aligned}$$

Therefore,

$$v'' = 0$$

and a satisfying $v(x)$ is $v(x) = x$.
Hence, one could choose

$$y_2(x) = v(x) \cdot x = x^2.$$

Variation of Parameters: We compute

$$W(x) = \det \begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} = x^2,$$

and note that $r(x) = 2x$. Thus,

$$\begin{aligned} y_p &= -y_1 \int \frac{r(x)y_2}{W} dx + y_2 \int \frac{r(x)y_1}{W} dx \\ &= -x \int \frac{(2x) \cdot x^2}{x^2} dx + x^2 \int \frac{(2x) \cdot x}{x^2} dx \\ &= -x^3 + 2x^3 \\ &= x^3. \end{aligned}$$

4. (20 points) Note that for the second-order equation

$$(1-x^2)y'' - y = 0,$$

$x_0 = 0$ is an ordinary point. Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be the (general) power series solution centered at zero.

(1) Find the first four nonzero terms in the power series solution $y(x)$. Your answer may involve a_0, a_1.

Consider that

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + ...$$

The equation becomes

$$(1-x^2)(2a_2 + 6a_3 x + 12a_4 x^2 + ...) - (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...) = 0.$$

Reorganizing according to the powers of x:

$$(2a_2 - a_0) + (6a_3 - a_1)x + (12a_4 - 2a_2 - a_2)x^2 + ... = 0.$$

Therefore,

$$\begin{aligned} a_2 &= \frac{1}{2}a_0, \\ a_3 &= \frac{1}{6}a_1, \\ a_4 &= \frac{1}{4}a_2 = \frac{1}{8}a_0, \\ &\cdots \end{aligned}$$

and

$$y(x) = a_0 + a_1 x + \frac{1}{2}a_0 x^2 + \frac{1}{6}a_1 x^3 + ...$$

(2) What is a lower bound for the radius of convergence of the series $y(x)$.

Ans. 1. Note that the roots of $1 - x^2$ are ± 1, both has distance to the origin at which the series solution is centered, then by a theorem in class.

5. (20 points) Suppose that, under natural conditions, the population of fish in a pond obeys the model

$$\dot{p} = p(4 - p).$$

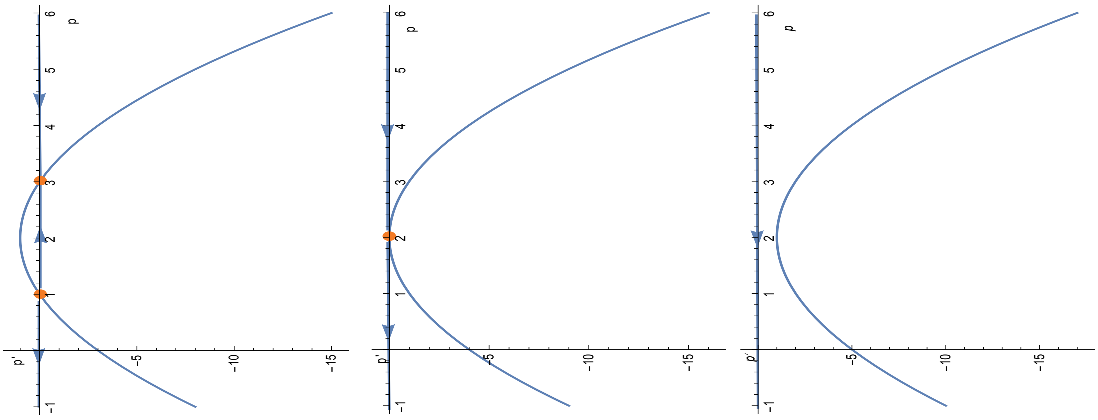
Now, suppose that people start harvesting continuously from the pond at a *constant* rate c. (Assume that the units are consistent.)

(1) Establish a model (differential equation) which describes the population of fish in the pond during harvesting.

Since the harvesting rate is a constant, we consider the model

$$\dot{p} = p(4 - p) - c.$$

(2) For $c = 3, 4, 5$ each, use the *phase-line* analysis to find the *equilibria* (if any) of your equation and *their stability*. Which value(s) of c among 3,4 and 5 would you choose if sustainability is favored?



(A) $c = 3$ (B) $c = 4$ (C) $c = 5$

(A): $c = 3$. Equation is

$$\dot{p} = -p(p - 4) - 3 = -(p - 1)(p - 3).$$

Hence the equilibrium solutions are $p = 1$ and $p = 3$, which are respectively unstable and asymptotically stable.
(B): $c = 4$. Equation is

$$\dot{p} = -(p - 2)^2.$$

Thus, $p = 2$ is an equilibrium, but unstable.
(C): $c = 5$. Equation is

$$\dot{p} = -(p - 2)^2 - 1.$$

Since the right hand side is always negative, the autonomous equation has no equilibrium.

If provided the three choices $c = 3, 4, 5$ only, one would certainly choose $c = 3$ for sustainability.

Bonus. (10 points) In the theory of PDEs, *Darboux integrability* is a property that certain PDEs have which enables one to find solutions using ODE techniques. The Norwegian mathematician Sophus Lie, in the early 1880s, realized that the f-Gordon equation

$$\frac{\partial^2 z}{\partial x \partial y} = f(z),$$

is Darboux integrable if and only if the single-variable function $f(x)$ satisfies the nonlinear second order ODE

$$(*) \qquad f'' f - (f')^2 = 0.$$

Now, try to find the general solutions of the equation $(*)$.

Solutions 1: The equation can be easily transformed into

$$\frac{f''}{f'} = \frac{f'}{f}.$$

Integrating on both sides gives

$$\ln |f'| = \ln |f| + C,$$

and hence

$$f' = Af$$

for some constant A. Thus,

$$f = Be^{Ax}$$

where A, B are constants.
(Note: This solving technique is in the spirit of separation of variables. As one may introduce $z = f'$, the original equation can then be written as

$$z' f = z f',$$

which is in a separable form.)

Solution 2: Dividing the equation by $(f')^2$ gives

$$\frac{f'' f' - (f')^2}{(f')^2} = 0,$$

which is simply

$$-\left(\frac{f}{f'}\right)' = 0.$$

Thus solutions must satisfy

$$f'/f = A,$$

for some constant C. Hence,

$$f = Be^{Ax}.$$

Solution 3 - Idea: (by a student in class) There is yet another possibility for obtaining the correct solutions. First observe that $f = e^x$ is a solution, then use the idea of reduction of orders by asking when is $v(x)e^x$ a solution. Calculation gives that $v(x)$ satisfies the same equation as above, hence one may choose $v(x) = e^x$, leading to solutions of the form $f = e^{nx}$ if carried out iteratively. One could now guess that solutions take the form $f(x) = Be^{Ax}$, though a disadvantage lurks behind this approach, i.e., it is not clear that the general solutions are all in this form.