EXAM II SOLUTIONS, MATH 353 FALL 2016

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. This test contains **8 pages** and **4 questions**. You have **50 minutes** to answer all the questions. *Good Luck* !

Laplace Transforms:

$$\begin{aligned} \mathcal{L}\{1\} &= \frac{1}{s} & \mathcal{L}\{\sin at\} &= \frac{a}{a^2+s^2} \\ \mathcal{L}\{t\} &= \frac{1}{s^2} & \mathcal{L}\{\cos at\} &= \frac{s}{a^2+s^2} \\ \mathcal{L}\{e^{at}f(t)\} &= F(s-a) & \mathcal{L}\{u_c(t)f(t-c)\} &= e^{-cs}F(s) \\ \mathcal{L}\{tf(t)\} &= -\frac{d}{ds}F(s) & \mathcal{L}\{f'(t)\} &= -f(0)+sF(s) \\ \mathcal{L}\{\delta(t-c)\} &= e^{-cs} & \mathcal{L}\{(f*g)(t)\} &= F(s)G(s) \end{aligned}$$

Question	Max. Points	Score
1	25	
2	25	
3	20	
4	30	
Total	100	

Question 1. (25 points)

(a)(5 points) Let $f(t), g(t)$ be functions defined on $[0, \infty)$, write down the **definition** of the *convolution* between f and g:

$$(f * g)(t) := \int_0^t f(t-\tau)g(\tau)d\tau$$

(b)(5 points) List **two** properties of the convolution *other than* $\mathcal{L}\{(f*g)(t)\} = F(s)G(s)$.

- $(c_1 f_1 + c_2 f_2) * g = c_1(f_1 * g) + c_2(f_2 * g)$
- $f * g = g * f$
- $f * (g * h) = (f * g) * h$

(c)(15 points) Using the method of Laplace transform, find a funcion $y(t)$ which satisfies

$$y(t) = e^t \left[1 + \int_0^t e^{-\tau} y(\tau) d\tau\right].$$

Note that the right hand side of the equation above is simply

$$e^t + \int_0^t e^{t-\tau} y(\tau) d\tau.$$

Application of the Laplace transform to both sides of the equation and using the formula $\mathcal{L}\{(f * g)(t)\} = F(s)G(s)$ yields

$$\begin{aligned} \mathcal{L}\{y(t)\} &= \mathcal{L}\{e^t\} + \mathcal{L}\left\{\int_0^t e^{t-\tau} y(\tau) d\tau\right\} \\ &= \frac{1}{s-1} + \mathcal{L}\{e^t\}\mathcal{L}\{y(t)\}. \end{aligned}$$

Denoting $\mathcal{L}\{y(t)\}$ as $Y(s)$, we have

$$Y(s) = \frac{1}{s-1} + \frac{1}{s-1} Y(s).$$

From this, one easily solves

$$Y(s) = \frac{1}{s-2}.$$

By the first shifting formula $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$, one concludes that

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^{2t}.$$

Question 2. (25 points)

(a)(5 points) Sketch the graph of the periodic function $f(x)$ satisfying

$$\begin{cases} f(x) = e^x, & -\pi \le x < \pi, \\ f(x) = f(x+2\pi), & x \in \mathbb{R}. \end{cases}$$

(*Sketch omitted.*)

(b)(10 points) Show by definition that the Fourier series of $f(x)$ is

$$g(x) = \frac{\sinh\pi}{\pi}\Big(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx - n\sin nx)\Big).$$

Hint:

$$\int e^x \sin nx dx = \frac{1}{n^2+1}e^x(\sin nx - n\cos nx),$$
$$\int e^x \cos nx dx = \frac{1}{n^2+1}e^x(\cos nx + n\sin nx).$$

Taking $2\pi(=2L)$ as the period, the Fourier series takes the form

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx).$$

By the Euler-Fourier formulae, we have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi}(e^{\pi} - e^{-\pi}) = \frac{2}{\pi}\sinh(\pi), \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx dx = \frac{1}{\pi}\frac{1}{n^2+1}e^x(\cos nx + n\sin nx)\Big|_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\frac{1}{n^2+1}[e^{\pi}(-1)^n - e^{-\pi}(-1)^n] \\ &= \frac{1}{\pi}\frac{2}{n^2+1}(-1)^n\sinh(\pi), \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx dx = \frac{1}{\pi}\frac{1}{n^2+1}e^x(\sin nx - n\cos nx)\Big|_{-\pi}^{\pi} \\ &= -\frac{1}{\pi}\frac{n}{n^2+1}[e^{\pi}(-1)^n - e^{-\pi}(-1)^n], \\ &= -\frac{1}{\pi}\frac{2n}{n^2+1}(-1)^n\sinh(\pi). \end{aligned}$$

Therefore,

$$\begin{aligned} g(x) &= \frac{\sinh(\pi)}{\pi} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{n^2+1}(-1)^n\sinh(\pi)\cos nx - \frac{2n}{n^2+1}(-1)^n\sinh(\pi)\sin nx\right) \\ &= \frac{\sinh(\pi)}{\pi}\Big(1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(\cos nx - n\sin nx)\Big), \end{aligned}$$

as desired.

(c)(10 points) What is the value of $g(\pi)$? What is the limit of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

Show your reasoning.

By the Fourier convergence theorem,

$$g(\pi) = \frac{1}{2}(f(\pi^-) + f(\pi^+)) = \frac{1}{2}(e^{\pi} + e^{-\pi}) = \cosh \pi.$$

On the other hand, by its very expression

$$g(\pi) = \frac{\sinh \pi}{\pi}\left(1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2}\right).$$

Comparing the two equalities above, we have

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}(\pi \coth \pi - 1).$$

Question 3. (20 points)

(a)(10 points) Solve the two-point boundary value problem for $X(x)$:

$$\begin{cases} X'' + \lambda X = 0, \quad 0 < x < \pi, \\ X(0) = X(\pi) = 0 \end{cases}$$

by finding all the eigenvalues/eigenfunctions.
Note: You may assume that all the eigenvalues are real.

Assuming that the eigenvalues are real, there are three cases: $\lambda = 0, > 0, < 0$.

Case 1: $\lambda = 0$. We have $X = ax + b$. The boundary conditions impose

$$b = 0, \quad a\pi + b = 0.$$

Hence, $a = b = 0$; and $\lambda = 0$ is not an eigenvalue.

Case 2: $\lambda = \mu^2 > 0$. This implies

$$X = c_1 \cos \mu x + c_2 \sin \mu x.$$

Thus,

$$X(0) = c_1 = 0, \quad X(\pi) = c_1 \cos \mu\pi + c_2 \sin \mu\pi = 0.$$

It follows that μ is a positive integer. Indeed, we have obtained a sequence of eigenvalues (indexed by k) $\lambda_k = \mu_k^2$, where $\mu_k = k > 0$ are integers. The corresponding eigenfunctions can be chosen to be $X_k(x) = \sin kx$.

Case 3: $\lambda = -\mu^2 < 0$. We have

$$X = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$

Here, c_1, c_2 must satisfy

$$X(0) = c_1 + c_2 = 0, \quad X(\pi) = c_1 e^{\mu\pi} + c_2 e^{-\mu\pi} = 0.$$

Regardless of the value of μ, one must have $c_1 = c_2 = 0$. No eigenvalues in this case.

(b)(10 points) For which values of $\alpha > 0$ does the two-point boundary value problem

$$\begin{cases} y'' + 4y = \cos x, \\ y(0) = y(\alpha) = 0. \end{cases}$$

have

(i) a **unique** solution;

(ii) **infinitely many** solutions.

Hint: The formula $\cos 2\alpha = 2\cos^2\alpha - 1$ *may be useful for part (ii).*

A particular solution of $y'' + 4y = \cos x$ is

$$y_p = \frac{1}{3}\cos x,$$

which can be easily found with the method of undetermined coefficients. Hence, the general solution of the equation $y'' + 4y = \cos x$ takes the form

$$y = \frac{1}{3}\cos x + c_1 \cos 2x + c_2 \sin 2x.$$

Considering the boundary conditions, we have

$$\begin{cases} y(0) = \dfrac{1}{3} + c_1 = 0, \\ \\ y(\alpha) = \dfrac{1}{3}\cos\alpha + c_1\cos 2\alpha + c_2 \sin 2\alpha = 0. \end{cases}$$

Written in matrix form, these conditions are

$$\begin{pmatrix} 1 & 0 \\ \cos 2\alpha & \sin 2\alpha \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 \\ \cos\alpha \end{pmatrix}.$$

(i) The original boundary problem has a unique solution if and only if the coefficient matrix

$$\begin{pmatrix} 1 & 0 \\ \cos 2\alpha & \sin 2\alpha \end{pmatrix}$$

is invertible. Equivalently, $\sin 2\alpha \neq 0$. For positive α's, this is saying that $\alpha \neq \frac{k\pi}{2}$, for any integer $k > 0$.

(ii) The boundary problem has infinitely many solutions if and only if the coefficient matrix is degenerate, and the vector $-\frac{1}{3}(1, \cos\alpha)^T$ lies in the column space of the matrix.

The degeneracy of the matrix imposes that $\sin 2\alpha = 0$, that is, $\alpha = \frac{k\pi}{2}$ for $k > 0$ being an integer. The condition that $-\frac{1}{3}(1, \cos\alpha)^T$ lies in the column space then requires that $(1, \cos 2\alpha)$ is parallel to $(1, \cos\alpha)$, as vectors in $\mathbb{R}^2$. As a result, we need

$$\cos 2\alpha = \cos\alpha$$

to hold, which is

$$2\cos^2\alpha - \cos\alpha - 1 = (2\cos\alpha + 1)(\cos\alpha - 1) = 0.$$

In order for the solution to be compatible with $\sin 2\alpha = 0$, we must have $\cos\alpha = 1$. Therefore,

$$\alpha = 2k\pi, \quad k = 1, 2, 3, ...$$

Question 4. (30 points)

Consider the initial boundary value problem

$$\begin{cases} u_{xx} = u_t - \sin x, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = 2\sin 2x + 5\sin 5x, & 0 \le x \le \pi. \end{cases}$$

(a)(10 points) Let $w(x)$ be a **steady-state** solution associated to the problem above, that is, $w(x)$ satisfies the differential equation and the boundary conditions. Find $w(x)$.

The equation for $w(x)$ is

$$w'' = -\sin x.$$

Hence, an obvious particular solution for $w(x)$ is

$$w(x) = \sin x,$$

which already satisfies the boundary conditions

$$w(0) = w(\pi) = 0.$$

(b)(5 points) Let $v(x,t) = u(x,t) - w(x)$. Write down the initial boundary value problem satisfied by $v(x,t)$.

It is a straight-forward verification that $v(x,t)$ satisfies

$$\begin{cases} v_{xx} = v_t, & 0 < x < \pi, \quad t > 0, \\ v(0,t) = v(\pi,t) = 0, & t > 0 \\ v(x,0) = 2\sin 2x + 5\sin 5x - \sin x, & 0 \le x \le \pi. \end{cases}$$

In particular, note that the $(\sin x)$-term in the equation is cancelled, which is expected since, loosely speaking, it is a term of "nonhomogeneity".

(c)(15 points) Solve the problem you wrote down in part (b) using the method of separation of variables. Then conclude by writing down the solution $u(x,t)$ of the original problem.
Hint: Your answer for Question 3(a) may be helpful.

Suppose that $v(x,t) = X(x)T(t)$ is a solution of the boundary value problem for v, we must have

$$X''T = XT',$$

leading to

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda,$$

for some constant λ. This splits into two ODEs:

$$\begin{cases} X'' + \lambda X = 0, \\ T' + \lambda T = 0. \end{cases}$$

Furthermore, since $v(0,t) = X(0)T(t) = 0$, $v(\pi,t) = X(\pi)T(t) = 0$ for all $t > 0$, we have

$$X(0) = X(\pi) = 0.$$

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Consequently, $X(x)$ satisfies the two-point boundary value problem as given in Question 3(a). The eigenvalues are

$$\lambda_k = k^2,$$

with corresponding eigenfunctions chosen to be

$$X_k(x) = \sin kx,$$

for $k = 1, 2, 3, ...$

For a particular index k, the ODE for $T(t)$ is then

$$T' + \lambda_k T = 0.$$

One immediately solves for

$$T_k(t) = e^{-k^2 t}.$$

Hence, we obtain the functions

$$v_k(x,t) = X_k(x)T_k(t) = e^{-k^2 t} \sin kx.$$

In general, a superposition

$$v(x,t) = \sum_{k=1}^{\infty} b_k v_k(x,t) = \sum_{k=1}^{\infty} b_k e^{-k^2 t} \sin kx$$

solves the boundary value problem for v. To meet the initial condition, we let $t = 0$ in the superposition formula above, obtaining

$$\sum_{k=1}^{\infty} b_k \sin kx = 2 \sin 2x + 5 \sin 5x - \sin x.$$

This implies that

$$b_2 = 2, \quad b_5 = 5, \quad b_1 = -1; \qquad b_k = 0 \ (k \neq 1, 2, 5)$$

and

$$v(x,t) = -e^{-t} \sin x + 2e^{-4t} \sin 2x + 5e^{-25t} \sin 5x.$$

To conclude, the solution of the original IBVP (in $u(x,t)$) is

$$u(x,t) = v(x,t) + w(x) = (1 - e^{-t}) \sin x + 2e^{-4t} \sin 2x + 5e^{-25t} \sin 5x.$$