LECTURE 9: SERIES SOLUTIONS NEAR AN ORDINARY POINT I

In this lecture and the next two, we will learn series methods through an attempt to answer the following two questions:

- What is a series method and how to use it?
- Is there any caution or limitation of using series methods?

The answer to the first question involves the method of *successive differentiations* and the method of *undetermined coefficients*. We will see how each of them works by studying examples. The second question arises as one notices that when working with the series method, one is restricted to the "analytic category", that is, all the functions that we are dealing with are analytic near the point of consideration. In particular, this means that we have to pay special attention to the radii of convergence of the series which are derived in the series method. Another reason for the second question to arise is the presence of *singularities*, for instance, $x = 0$ for the equation $xy'' + y = 0$. Previously, we put all second order linear equations in the form $y'' + p(x)y' + q(x)y = g(x)$, and simply ruled out all the singularities from the domain of definition. Here, an application of the series method seem to give us a series solution near a singular point. The concern is, of course, the convergence of the series solution.

1. Successive Differentiation

Example 1. Consider the initial value problem

$$y'' + y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Assume that the solutions are analytic near $x = 0$. In other words, they have the Taylor expansion:

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + ...$$

Thus, given the initial value $y(0) = 1, y'(0) = 0$, all we need to do to solve the equation is to determine all the derivatives $y^{(n)}(0)$ in the Taylor expansion. In fact, they can be determined successively.

By the equation, we have

$$y''(0) = -y(0) = -1.$$

Moreover,

$$y^{(3)}(0) = (y'')'(0) = (-y)'(0) = -y'(0) = 0,$$

and

$$y^{(4)}(0) = (y'')''(0) = (-y)''(0) = -(-y)(0) = y(0) = 1.$$

See the pattern? We have

$$y^{(4k+\ell)}(0) = \begin{cases} 1, & \ell = 0 \\ 0, & \ell = 1 \\ -1, & \ell = 2 \\ 0, & \ell = 3 \end{cases}$$

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Therefore,

$$y(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + ...,$$

which is exactly the Taylor expansion of $\cos x$ at $x = 0$.

Example 2. Consider the initial value problem

$$y'' - xy' + x^2 y = 0, \qquad y(0) = 1, \quad y'(0) = 1.$$

As in the previous example, we assume that the solutions are analytic near $x = 0$ for now. Thus, we have the Taylor expansion:

$$y(x) = y(0) + y'(0)x + \frac{1}{2!}y''(0)x^2 + \frac{1}{3!}y'''(0)x^3 + ...,$$

in which the coefficients, the derivatives $y^{(k)}(0)$ are to be determined.

Now, by carrying out the successive derivatives, we obtain

$$\begin{aligned} y''(0) &= [xy' - x^2 y]_{x=0} = 0; \\ y'''(0) &= [xy' - x^2 y]'_{x=0} = [y' + xy'' - 2xy - x^2 y']_{x=0} = 1; \\ y^{(4)}(0) &= [y' + xy'' - 2xy - x^2 y']'_{x=0} = [2y'' + xy''' - 2y - 4xy' - x^2 y'']_{x=0} = -2. \end{aligned}$$

As you notice, it is harder to see a pattern in this example, and, as we progress, the expressions of the derivatives are getting longer. So, we simply stop at the first four terms:

$$y(x) = 1 + x + \frac{1}{3!}x^3 - \frac{2}{4!}x^4 + ...,$$

and instead, introduce a more convenient way to figure out the coefficients in the Taylor expansion: undetermined coefficients.

2. Undetermined Coefficients

The method of undetermined coefficients avoids computing successive derivatives. The idea is, still assume the analyticity of the solution, but let it take the form

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n + ... = \sum_{k=0}^{\infty} a_k x^k.$$

Plugging this expression in the equation, one should be able to obtain certain relations among the coefficients a_k. Let us see how this method works in the Example 2 in the previous section.

Example 2.(Revisited) Here for the initial value problem

$$y'' - xy' + x^2 y = 0, \qquad y(0) = 1, \quad y'(0) = 1,$$

we assume that its solution near $x = 0$ takes the form

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_n x^n + ... = \sum_{k=0}^{\infty} a_k x^k.$$

Note that the equation involves $y''(x)$ and $y'(x)$. We calculate:

$$y'(x) = (a_0 + a_1x + a_2x^2 + ...)' = a_1 + 2a_2x + 3a_3x^2 + ... = \sum_{k=1}^{\infty} ka_kx^{k-1},$$
$$y''(x) = (a_1 + 2a_2x + 3a_3x^2 + ...)' = 2a_2 + 3 \cdot 2a_3x + ... = \sum_{k=2}^{\infty} k(k-1)a_kx^{k-2}.$$

Plugging this in the equation, and shifting the indices to make the summands appear in the form x^k, we have

$$\begin{aligned}
0 &= y'' - xy' + x^2y \\
&= \sum_{k=2}^{\infty} k(k-1)a_kx^{k-2} - x\sum_{k=1}^{\infty} ka_kx^{k-1} + x^2\sum_{k=0}^{\infty} a_kx^k \\
&= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=1}^{\infty} ka_kx^k + \sum_{k=2}^{\infty} a_{k-2}x^k \\
&= 2a_2 + (6a_3 - a_1)x + \sum_{k=2}^{\infty}[(k+2)(k+1)a_{k+2} - ka_k + a_{k-2}]x^k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
a_2 &= 0, \\
a_3 &= \frac{1}{6}a_1, \\
a_{k+2} &= \frac{1}{(k+2)(k+1)}(ka_k - a_{k-2}), \qquad k \geq 2.
\end{aligned}$$

Under the initial condition $y(0) = 1, \quad y'(0) = 1$, we have $a_0 = a_1 = 1$, hence a_0, a_1, a_2, a_3 are known. Successively, by the third equality above, we could find $a_4, a_5, ...$ successively. Formulas like this are called *recurrence relations*, i.e., a relation which enables one compute all the items in an array recursively from a few initial items.

Up to now, we have only studied examples in which the initial values are at $x = 0$. Here is a variation of Example 2, by setting the initial value at $x = 1$. Of course, we still need to assume the analyticity of the solution near $x = 1$.

Example 2.(Initial Value at $x = 1$) Consider the initial value problem

$$y'' - xy' + x^2y = 0, \qquad y(1) = 1, \quad y'(1) = 1.$$

By the assumption, the solution is in the form

$$y(x) = b_0 + b_1(x-1) + b_2(x-1)^2 + ... + b_n(x-1)^n + ... = \sum_{k=0}^{\infty} b_k(x-1)^k.$$

Hence,

$$y'(x) = b_1 + 2b_2(x-1) + ... = \sum_{k=1}^{\infty} kb_k(x-1)^{k-1},$$
$$y''(x) = 2b_2 + 3 \cdot 2b_3(x-1) + ... = \sum_{k=2}^{\infty} k(k-1)b_k(x-1)^{k-2}.$$

Plugging this in the equation and noting that

$$x = (x-1)+1, \qquad x^2 = (x-1)^2 + 2(x-1) + 1,$$

we have

$$\begin{aligned} 0 &= y'' - ((x-1)+1)y' + ((x-1)^2 + 2(x-1)+1)y \\ &= \sum_{k=2}^{\infty} k(k-1)b_k(x-1)^{k-2} - ((x-1)+1)\sum_{k=1}^{\infty} kb_k(x-1)^{k-1} \\ &\qquad + ((x-1)^2 + 2(x-1)+1)\sum_{k=0}^{\infty} b_k(x-1)^k \\ &= \sum_{k=0}^{\infty}(k+2)(k+1)b_{k+2}(x-1)^k - \sum_{k=1}^{\infty} kb_k(x-1)^k - \sum_{k=0}^{\infty}(k+1)b_{k+1}(x-1)^k \\ &\qquad + \sum_{k=2}^{\infty} b_{k-2}(x-1)^k + 2\sum_{k=1}^{\infty} b_{k-1}(x-1)^k + \sum_{k=0}^{\infty} b_k(x-1)^k \\ &= (2b_2 - b_1 + b_0) + (6b_3 - 2b_2 + 2b_0)(x-1) \\ &\quad + \sum_{k=2}^{\infty}((k+2)(k+1)b_{k+2} - (k-1)b_k - (k+1)b_{k+1} + b_{k-2} + 2b_{k-1})(x-1)^k. \end{aligned}$$

(You could see here a relatively heavier usage of the shift of indices.)

Therefore,

$$\begin{aligned} 2b_2 - b_1 + b_0 &= 0, \\ 6b_3 - 2b_2 + 2b_0 &= 0, \\ (k+2)(k+1)b_{k+2} - (k-1)b_k - (k+1)b_{k+1} + b_{k-2} + 2b_{k-1} &= 0, \qquad k \geq 2. \end{aligned}$$

It is easy to see that this recursive relation can be used to compute all the coefficients b_k, given b_0, b_1.

We close this section by an example in which even a general formula for the recursive relations of the coefficients is hard to obtain.

Example 3. Consider the equation

$$y'' + \cos xy' + e^x y = 0.$$

We use the power series method to find solutions, assuming they are analytic near $x = 0$.
As before, we assume

$$y(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$$

Thus,

$$y'(x) = (a_0 + a_1x + a_2x^2 + ...)' = a_1 + 2a_2x + 3a_3x^2 + ... = \sum_{k=1}^{\infty} ka_kx^{k-1},$$

$$y''(x) = (a_1 + 2a_2x + 3a_3x^2 + ...)' = 2a_2 + 3 \cdot 2a_3x + ... = \sum_{k=2}^{\infty} k(k-1)a_kx^{k-2}.$$

Also note that

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - ...$$
$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$$

Plugging these expansions in the equation, and rearranging the terms according to the powers of x, we obtain

$$\begin{aligned} 0 &= (2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + ...) \\ &\quad + (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + ...)(a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + ...) \\ &\quad + (1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...)(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4...) \\ &= (2a_2 + a_1 + a_0) + (6a_3 + 2a_2 + a_1 + a_0)x + (12a_4 + \tfrac{1}{2}a_1 + 3a_3 + a_2 + \tfrac{1}{2}a_0)x^2 \\ &\quad + (20a_5 + 4a_4 + a_3 + \tfrac{1}{2}a_1 + \tfrac{1}{6}a_0)x^3 + ... \end{aligned}$$

Therefore,

$$\begin{aligned} a_2 &= -\frac{a_1 + a_0}{2}, \\ a_3 &= -\frac{2a_2 + a_1 + a_0}{6} = 0, \\ a_4 &= -\frac{a_1 + 2a_2 + a_0}{24} = 0, \\ a_5 &= -\frac{3a_1 + a_0}{120}, \\ &...... \end{aligned}$$

and

$$y(x) = a_0(1 - \frac{1}{2}x^2 - \frac{1}{120}x^5 + ...) + a_1(x - \frac{1}{2}x^2 - \frac{1}{40}x^5 + ...).$$

Here, we've specified the terms up to order five in a general solution of the given equation.

Remark. After seeing these examples, one could ask: (1) How could we assume that the solutions are analytic near the initial value? If it is analytic, can we quantify its radius of convergence rather than just saying "nearby x_0"? (2) When we plug the power series into the equations, what we did next was to group all the powers of x and equate the coefficients. Why are we allowed to do this?

Given the analyticity of the solutions and the functions in the equation, the answer for (2) is quite simple: By plugging in the power series, we equate the Taylor expansions of two analytic functions. Because the Taylor expansions of an analytic function is unique at a point, the coefficients in the expansion must agree.

The answer for (1) will be discussed in the next lecture.