LECTURE 8: REVIEW OF POWER SERIES

Let us remember the chart at the end of lecture 6. It was shown there that for the non-constant coefficient second order linear ODEs, we know how to solve the homogeneous equation completely from knowing one nonzero solution, and how to solve the non-homogeneous equation from knowing the fundamental solutions to the homogeneous equation. However, we did not know a general method for finding one solution for the homogeneous equation. This lecture and on, we are going to explore several methods which enable us to find solutions of linear equations. As you will see, these methods share some "guiding principles", which is to decompose the solution into pieces of certain "standard" functions, determine each piece individually, then put them together. A simple idea following this line of thought is viewing the solution as a power series.

1. Power Series

A series of the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots$$

where $a_0, a_1, \dots$ and x_0 are constants is called a *power series*.

Note that for any fixed value of x, the power series becomes an infinite series, and the power series is said to *converge* at $x = x_1$ if $a_0+a_1(x_1-x_0)+a_2(x_1-x_0)^2+a_3(x_1-x_0)^3+\dots$ converges. Of course, the power serious converges at $x = x_0$, but it may also be convergent in some interval containing x_0, even the entire $(-\infty, \infty)$. In the last two cases, the set of values of x for which the power series converges is called the *interval of convergence* of the series (yes, we are going to see that this set is indeed an interval).

In calculus, we have learned various tests by which we can tell whether a series converges. One frequently used test is called the "ratio test", which states that the series

$$c_0 + c_1 + c_2 + \dots$$

is *absolutely convergent*, that is,

$$\lim_{n\to\infty} (|c_0| + |c_1| + |c_2| + \dots + |c_n|) < \infty,$$

if

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = k < 1.$$

On the other hand, if

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = k > 1,$$

then the series

$$c_0 + c_1 + c_2 + \dots$$

diverges, meaning that

$$\lim_{n\to\infty} (c_0 + c_1 + \dots + c_n)$$

does not exist.

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One may ask, what if

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = 1?$$

The answer is indefinite. Consider the following three series, which all satisfy this above equality:

$$(1) \qquad 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}+...$$

$$(2) \qquad 1-\frac{1}{2}+\frac{1}{3}-...+(-1)^{n-1}\frac{1}{n}+...$$

$$(3) \qquad 1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{n^2}+...$$

Here, series (1) is divergent, series (2) is convergent but not absolutely convergent, series (3) is absolutely convergent (hence convergent).

Applying the ratio test to the power series

$$a_0+a_1(x-x_0)+a_2(x-x_0)^2+a_3(x-x_0)^3+...,$$

the three statements above become:

- If
$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = k<1,$$
then the power series
$$a_0+a_1(x-x_0)+a_2(x-x_0)^2+a_3(x-x_0)^3+...$$
converges absolutely.
- If
$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = k>1,$$
then the power series
$$a_0+a_1(x-x_0)+a_2(x-x_0)^2+a_3(x-x_0)^3+...$$
diverges.
- If
$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$$
it is indefinite in general whether
$$a_0+a_1(x-x_0)+a_2(x-x_0)^2+a_3(x-x_0)^3+...$$
converges or diverges.

As we can see, the limit of ratios $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ plays a crucial role here in determining: for which values of x the power series is convergent or divergent. With this in mind, let us define the *radius of convergence* of the power series $a_0+a_1(x-x_0)+a_2(x-x_0)^2+a_3(x-x_0)^3+...$ to be

$$\rho=\begin{cases}\infty, & \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0,\\ \left(\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|\right)^{-1}, & \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>0.\end{cases}$$

Therefore, corresponding back to the ratio test, we have that the power series is convergent on $(x_0 - \rho, x_0 + \rho)$, divergent on $(-\infty, x_0 - \rho) \cup (x_0 + \rho, \infty)$, and is indefinite (or dependent on the specific expression of the power series) at the end-points $x_0 - \rho$ and $x_0 + \rho$ when $\rho < \infty$. Now, we can argue that the set of x-values at which a power series is convergent forms an interval (open, closed or half-closed). Do you see why?

Examples.
(1) The series

$$1 + 1!x + 2!x^2 + 3!x^3 + ... + n!x^n + ...$$

converges only at $x = 0$ and diverges everywhere else, because

$$\lim_{n\to\infty} \left|\frac{(n+1)!}{n!}\right| = \lim_{n\to\infty} (n+1) = \infty.$$

Thus, the radius of convergence ρ equals to zero.

(2) The series

$$1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + ... + \frac{1}{n}x^n + ...$$

is convergent on the half-closed interval $[-1, 1)$ because on the one hand, its radius of convergence is

$$\rho = \Big(\lim_{n\to\infty} \left|\frac{1/(n+1)}{1/n}\right|\Big)^{-1} = 1,$$

on the other hand, it is convergent at $x = -1$ but divergent at $x = 1$.

(3) The series

$$1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ... + \frac{1}{n!}x^n + ...$$

is convergent on the entire $(-\infty, \infty)$ because its radius of convergence is ∞ since

$$\lim_{n\to\infty} \left|\frac{1/(n+1)!}{1/n!}\right| = 0.$$

*(4) The series

$$1 + x^2 + \frac{1}{2}x^4 + \frac{1}{3}x^6 + ... + \frac{1}{n}x^{2n} + ...$$

has the interval of convergence $(-1, 1)$, but it seems that the ratio test does not directly tell us this. In fact, if we calculate

$$\frac{a_{n+1}}{a_n},$$

it equals zero when n is even and ∞ when n is odd, hence the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ does not exist (not even equal to ∞). However, there is a way to use the ratio test indirectly. Take $z = x^2$, of course, $z \geq 0$, the power series then becomes

$$1 + z + \frac{1}{2}z^2 + ...$$

which has the convergence interval $(-1, 1)$. Note that $z \geq 0$. Therefore, in x, the interval of convergence consists those x satisfying $z = x^2 \in [0, 1)$, i.e., $x \in (-1, 1)$. Side Note: If your are familiar with the Taylor expansions of $\sin x$ and $\cos x$, and are trying to use the ratio test determine their radii of convergence, you'd probably think of the trick used in this exercise.

Theorem. The power series

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + ...$$

defines a continuous function on the open interval $(x_0 - \rho, x_0 + \rho)$, where ρ is its radius of convergence.

The following theorem tells us what the derivative of a power series is in its interval of convergence.

Theorem. Under the same conditions as in the preceding theorem, and let

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + ..., \qquad x_0 - \rho < x < x_0 + \rho.$$

Then

$$f'(x) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + ..., \qquad x_0 - \rho < x < x_0 + \rho.$$

We are not going to prove these theorems in this course, but instead ask a natural question that follows the theorems: if $f(x)$ is such a continuous function, how do the coefficients in the power series relate to $f(x)$? Of course, you may be thinking of the Taylor series. So let us derive the formulas for all the coefficients a_i.

First, let us assume that on $(x_0 - \rho, x_0 + \rho)$,

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + ...$$

Take $x = x_0$ and immediately we have

$$f(x_0) = a_0.$$

Take $x = x_0$ in the expression of $f'(x)$, and we have

$$f'(x_0) = a_1.$$

Now apply the second theorem above to the function $f'(x)$, we get

$$f''(x) = 2!a_2 + 3 \cdot 2a_3(x - x_0) + ..., \qquad x_0 - \rho < x < x_0 + \rho.$$

Again, evaluating this expression at x_0 gives

$$f''(x_0) = 2!a_2.$$

Now you may have seen the pattern, the expression for $f^{(n)}(x)$ is

$$f^{(n)}(x) = n!a_n + (n + 1)n...2a_{n+1}(x - x_0) + ..., \quad x_0 - \rho < x < x_0 + \rho.$$

Hence, for the same reason as before,

$$f^{(n)}(x_0) = n!a_n.$$

To summarize, if the power series

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + ...$$

is convergent on $(x_0 - \rho, x_0 + \rho)$, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

And the power series

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + ...$$

is called the *Taylor expansion* of $f(x)$ *in a neighborhood of* x_0.

2. Shift of Summation Index

Using the summation notation, $a_1 + ... + a_n$ can be conveniently written as $\sum_{i=1}^{n} a_i$. Of course, there is no restriction to choosing the symbol i as the index. In fact, writing the sum as $\sum_{j=1}^{n} a_j$ is also valid. The indices i and j in the summation expressions are called "dummy indices" because they are merely place-holders and do not appear in the result of the sum. Essentially, we do not even need to have j to start strictly from 1 and end at n. The expression $\sum_{k=15}^{n+15} a_{k-15}$ represents exactly the same summation $a_1 + a_2 + ... + a_n$, because, despite a different range of the dummy index, what's being summed up remains unchanged.

The observation in the previous paragraph applies to the case of power series. In the power series $a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + ...$, the form of a general term is $a_n(x - x_0)^n$, hence the series can be written as $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ representing the limit $\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x - x_0)^n$. Again, there is no restriction in our choice of the symbol n as the index, nor there is any restriction to choosing zero as the starting value of the index. The expressions $\sum_{n=2}^{\infty} a_{n-2}(x - x_0)^{n-2}$ and $\sum_{k=10}^{-\infty} a_{10-k}(x - x_0)^{10-k}$ both represent the same power series.

Through the following exercise, we will learn how to shift the index to obtain certain desired forms of the summation. One reason for this is that sometimes, we need to compare two or more power series, so it would be convenient if we are looking at the corresponding terms of the same power of x.

Exercise. Put the following power series in the summation form in which the power of $(x - x_0)$ is the dummy index n:

$$\sum_{n=1}^{\infty} na_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n.$$

Answer. Letting $m = n - 1$ in the first sum, we have that m starts from zero and goes to infinity. Similarly, let $k = n + 1$ in the second sum. Therefore,

$$\begin{aligned}
&\sum_{n=1}^{\infty} na_n x^{n-1} + 2x \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{m=0}^{\infty} (m + 1)a_{m+1} x^m + 2 \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{m=0}^{\infty} (m + 1)a_{m+1} x^m + 2 \sum_{k=1}^{\infty} a_{k-1} x^k.
\end{aligned}$$

Now since m, k are dummy indices, we could replace them by n, obtaining

$$\begin{aligned}&\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + 2\sum_{n=1}^{\infty}a_{n-1}x^n\\ =&a_1 + \sum_{n=1}^{\infty}((n+1)a_{n+1} + 2a_{n-1})x^n.\end{aligned}$$

3. Appendix: The Notion of Convergence

When we say that a sequence $\{a_n\}$ tends to 1, or *converges* to 1, what do we exactly mean? You can say, these numbers a_n "approach" 1 as n gets larger. But, again, what does "approach" mean exactly? If you think about this for a while, you will perhaps notice that "approaching" implies that one could put a "window" near the value 1, say the open interval $(0.9, 1.1)$, and there exists a moment after which all the a_n's are within this "window". Moreover, one could set this "window" to be however small he likes, and the moment being described always exists. It is probably because of this intuition, we have the formal definition for $\{a_n\}$ to be convergent, which lies at the base of the subject of mathematical analysis:

Definition. An infinite sequence of real numbers $\{a_n\}_{n=0}^{\infty}$ is said to be converging to a if for any positive number $\epsilon > 0$ (i.e., half the size of our "window"), there exists a positiver integer N (i.e., the described "moment"), such that for all $n > N$, $|a_n - a| < \epsilon$ (i.e., all the future numbers in the sequence lie in the "window").

Now, using this notion of convergence, we prove the first two claims of the ratio test for infinite series. To be precise, we prove the claims for special values of k's just to make the argument easier to understand. First, note that an infinite series $\sum_{i=1}^{\infty} a_n$ is equivalent to the sequence $\{s_n\}_{n=1}^{\infty}$ where $s_n = \sum_{i=1}^{n} a_i$ is the sum of the first n terms in the infinite series. For all n, such s_n are called partial sums of the infinite series.

Statement 1: Given an infinite series

$$a_0 + a_1 + ... + a_n + ...,$$

if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = k < 1$, then the series converges absolutely.

Proof for the special case $k = 0.8$: Because it is given $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8$, if we set the window around 0.8 to be $(0.7, 0.9)$, then there exists a positive integer N such that for all $n > N$, we have $0.7 < \left|\frac{a_{n+1}}{a_n}\right| < 0.9$. Now the infinite series $s = |a_0| + |a_1| + ... + |a_n| + ...$ can be viewed has having two parts, one is

$$s_N = |a_0| + ... + |a_N|,$$

which is clearly a finite number; the other is

$$s - s_N = |a_{N+1}| + ... + |a_{2N}| + ...$$

Now note that the ratio between two successive terms starting from index $N+1$ is less than 0.9, which means,

$$|a_{N+k}| = \frac{|a_{N+k}|}{|a_{N+k-1}|}|a_{N+k-1}| = \frac{|a_{N+k}|}{|a_{N+k-1}|}\frac{|a_{N+k-1}|}{|a_{N+k-2}|}|a_{N+k-2}|$$
$$= ... = \frac{|a_{N+k}|}{|a_{N+k-1}|}\frac{|a_{N+k-1}|}{|a_{N+k-2}|}...\frac{|a_{N+1}|}{|a_N|}|a_N| < 0.9^k|a_N|.$$

Therefore,

$$s - s_N < (0.9 + 0.9^2 + 0.9^3 + ...)|a_N| = \frac{1}{1-0.9}|a_N| = 10|a_N|,$$

which is finite. Combining with the fact that s_N is finite, we know that the sum $|a_0| + |a_1| + ...$ has to be finite. A theorem says, if a sequence of numbers is non-decreasing and bounded from above by a finite number, then it converges. Now our sequence $\{|a_0| + ... + |a_n|\}_{n=0}^{\infty}$ is non-decreasing and bounded by a finite number. Therefore, it converges, i.e., the original series converges absolutely.

Statement 2: Given an infinite series

$$a_0 + a_1 + ... + a_n + ...,$$

if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = k > 1$, then the series diverges.
Proof for the special case $k = 1.2$: For the same reason as before, we can set a window around 1.2 to be $(1.1, 1.3)$, and there exists a positive integer N such that for all $n > N$, $1.1 < \left|\frac{a_{n+1}}{a_n}\right| < 1.3$. And we could assume without loss of generality that $a_N \neq 0$. Therefore,

$$|a_{N+k}| = \frac{|a_{N+k}|}{|a_{N+k-1}|}|a_{N+k-1}| = \frac{|a_{N+k}|}{|a_{N+k-1}|}\frac{|a_{N+k-1}|}{|a_{N+k-2}|}|a_{N+k-2}|$$
$$= ... = \frac{|a_{N+k}|}{|a_{N+k-1}|}\frac{|a_{N+k-1}|}{|a_{N+k-2}|}...\frac{|a_{N+1}|}{|a_N|}|a_N| > 1.1^k|a_N|.$$

To finish the proof, again, we cite a theorem: if the series $a_0 + a_1 + ... + a_n + ...$ is convergent, then $a_k \to 0$ as $k \to \infty$. In our case, $\{a_n\}$ can never tend to zero because $|a_{N+k}| > 1.1^k|a_N| > |a_N|$ for any positive integer k. Therefore, the original series is divergent.