LECTURE 7: PROBLEM SESSION 1

1. Topics Outline

1.1. **First Order ODEs.**

1.**Linear Equations**: Integrating factors. Idea: For the equation $y' + p(x)y + q(x) = 0$, find a function $\mu(x)$, such that whenever $y(x)$ is a solution to the equation, $z(x) = \mu(x)y(x)$ solves an equation of the form $z' + \tilde{q}(x) = 0$.

2.**Separable Equations**: Equations which can be put in the form $p(x) + q(y)\frac{dy}{dx} = 0$. Often, we obtain implicit solutions for separable equations.

3.**Solving non-separable equations via separable ones**: e.g. consider the equation $\frac{dy}{dx} = \frac{x^2+xy+y^2}{xy}$, and use the substitution $z = \frac{y}{x}$.

4.**Autonomous Equations**: Vector field plot and phase-line analysis. Equilibrium and stability.

5.**Existence & Uniqueness Theorem**: Statement of the Theorem. Note: theorem is about initial value problems; the conclusion is local, nearby our initial value.

6.**Modeling**: Keys are, identify the unknown function, and use the relations that can be derived from the statement of the problem to establish equations.

7.**Exact Equations**: Test for being exact. If exact, how to find solutions? If not exact, is there any test that can tell us whether we can multiply the entire equation by some function and make the equation exact?

1.2. **Second Order Linear ODE.**

1.**Constant Coeff., Homogeneous**: Characteristic Polynomial.

2.**Constant Coeff., Non-homogeneous**: Undetermined Coefficients; or Variation of Parameters.

3.**Non-constant Coeff., Homogeneous**: One non-zero solution known, find another solution (linearly independent from the one already known) by Reduction of Order.

4.**Non-constant Coeff., Non-homogeneous**: From knowing a fundamental set of solutions of the underlying homogeneous equation to a particular solution of the non-homogeneous equation—Variation of Parameters.

2. Exercises

1. Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{3 + 2x + e^x}{y^4}.$$

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Solution. Note that this equation is separable. In fact, it can be put in the standard form

$$-(3+2x+e^x)+y^4\frac{dy}{dx}=0,$$

which, by realizing the antiderivatives of the left hand side, is just

$$\frac{d}{dx}(-3x-x^2-e^x+\frac{1}{5}y^5)=0.$$

Therefore, the general solution of the differential equation is, implicitly,

$$-3x-x^2-e^x+\frac{1}{5}y^5=C,$$

for arbitrary constant C. Particularly in our case, the solution can be expressed explicitly as

$$y(x)=(5(C+3x+x^2+e^x))^{\frac{1}{5}}.$$

2. Suppose that y_1 and y_2 are two solutions of the equation

$$y'=\frac{t^2y^3}{y^2-1},$$

and that they are different near the point $x=0$. Is it possible that $y_1(0)=y_2(0)=2$? Justify your answer.

Solution. We check the statement by checking the conditions in the Existence and Uniqueness theorem. First, we calculate

$$\frac{\partial}{\partial y}\Big(\frac{t^2y^3}{y^2-1}\Big)=\frac{3t^2y^2}{y^2-1}-\frac{2t^2y^4}{(y^2-1)^2}=\frac{t^2y^2(y^2-3)}{(y^2-1)^2}.$$

Observe that the functions $\frac{t^2y^3}{y^2-1}$ and $\frac{\partial}{\partial y}\big(\frac{t^2y^3}{y^2-1}\big)$ are continuous within the rectangle $(-1,1)\times(1,3)$ which contains the initial value $(0,2)$. This means that solution exists and is unique near $x=0$, which contradicts the description of y_1 and y_2. Therefore, y_1 and y_2, being different near $x=0$, cannot share the same initial value.

3. Solve the initial value problem

$$t^3y'+3t^2y=\cos t,\qquad y(\tfrac{\pi}{2})=0.$$

Solution. Here we present three different ways to solve this exercise.
(1) Observe that the left hand side is just the t-derivative of the expression t^3y, and immediately, the equation becomes

$$\frac{d}{dt}(t^3y)=\cos t.$$

Thus, we have general solutions

$$t^3y=\sin t+C.$$

The initial condition tells us that $C=-1$. Hence, the solutions is

$$y=\frac{\sin t-1}{t^3}.$$

(Certainly, there is no problem with dividing by t here. If we plug $t=0$ in the original equation, we get 0=1! Hence $t=0$ is not in the domain of definition.)

(2) Another approach can be rewriting the equation as:

$$(-\cos t + 3t^2 y) + t^3 \frac{dy}{dt} = 0.$$

And we realize that this equation is exact, because

$$\frac{\partial M}{\partial y} = 3t^2 = \frac{\partial N}{\partial t}.$$

Hence,

$$F(t, y) = \int M(t, y)dt + h(y) = \int (-\cos t + 3t^2 y)dt + h(y) = -\sin t + t^3 y + h(y).$$

Moreover,

$$\frac{\partial F}{\partial y} = t^3 + h'(y) = N(t, y) = t^3.$$

Therefore, we can choose $h(y) = 0$ and obtain

$$F(t, y) = t^3 y - \sin t = C$$

as the general solution. Again, the initial value gives $C = -1$. Hence $y = \frac{\sin t - 1}{t^3}$ is the solution.

(3) A third approach is by observing that the equation is linear and use the method of integrating factors. Suppose that $\mu(x)$ is a desired integrating factor, then $z(x) = \mu(x)y(x)$ satisfies the equation

$$\begin{aligned} z' &= \mu' y + \mu y' \\ &= \frac{\mu'}{\mu} z + \mu \frac{\cos t - 3t^2 y}{t^3} \\ &= \frac{\mu'}{\mu} z - \frac{3}{t} z + \frac{\cos t}{t^3} \mu. \end{aligned}$$

As the idea of integrating factor suggests, we need $\frac{\mu'}{\mu} - \frac{3}{z} = 0$, and one choice is evidently $\mu(x) = t^3$. Thus, with this choice of μ,

$$z' = \cos t.$$

Therefore,

$$z = \sin t + C,$$

and

$$y(x) = \frac{1}{\mu} z = \frac{\sin t + C}{t^3}.$$

Again, $C = -1$ by the initial value.

4. Find the general solutions of the differential equation

$$x^3 + y + 1 + (x - 2e^y - 2)\frac{dy}{dx} = 0.$$

Solution. Observe that this equation is exact, since

$$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}.$$

Thus,

$$F(x,y) = \int M(x,y)dx + h(y) = \frac{1}{4}x^4 + xy + x + h(y).$$

On the other hand,

$$\frac{\partial}{\partial y}F(x,y) = x + h'(y) = N(x,y) = x - 2e^y - 2.$$

This tells us that

$$h'(y) = -2e^y - 2.$$

Hence, a choice of the function $h(y)$ is

$$h(y) = -2e^y - 2y.$$

Now we obtain the general solution of the original equation, in the implicit form:

$$F(x,y) = \frac{1}{4}x^4 + xy + x - 2e^y - 2y = C,$$

for arbitrary constant C.

5. Given that the function $y_1(x) = x$ is a solution of the following equation

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \qquad x > 0.$$

Find the general solution of this equation.

Solution. We consider using the method of reduction of order. Supposing that $v(x)y_1(x) = v(x)x$ also satisfies the equation, we have

$$\begin{aligned} 0 =& x^2(vx)'' - x(x+2)(vx)' + (x+2)(vx) \\ =& x^2(v''x + 2v') - x(x+2)(v'x + v) + (x+2)vx \\ =& x^3v'' - x^3v'. \end{aligned}$$

Because it is assumed that $x > 0$, we have

$$v'' = v'.$$

Even without substituting $z = v'$ and solve for $z' = z$, an obvious choice for v is $v(x) = e^x$. Hence, the general solution for the original equation is

$$c_1x + c_2xe^x,$$

for arbitrary constants c_1, c_2.

6. Given that the function $y_h = c_1x^{-3} + c_2x^2$ is the general solution of the homogeneous differential equation

$$x^2y'' + 2xy' - 6y = 0,$$

find the general solution of the non-homogeneous equation

$$x^2y'' + 2xy' - 6y = x^2.$$

Solution. We realize that the variation of parameters is the technique to use to solve this exercise. In fact, we know already that a fundamental set of solutions of the underlying homogeneous equation is $\{x^{-3}, x^2\}$. Thus, the method of variation of parameters tells us that we can choose functions $v_1(x), v_2(x)$ such that $v_1(x)x^{-3} + v_2(x)x^2$ solves the non-homogeneous equation. Moreover, we have the formula (which, in my opinion, might be easier to remember than the formula that directly computes v_1 and v_2):

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ g(x) \end{pmatrix} = \frac{1}{W(x^{-3}, x^2)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Be careful, in our case, $g(x) = 1$ instead of x^2! And we have, $W(y_1, y_2) = 5x^{-2}$ and

$$v_1' = -\frac{1}{W(y_1, y_2)} y_2 = -\frac{1}{5} x^4,$$
$$v_2' = \frac{1}{W(y_1, y_2)} y_1 = \frac{1}{5} x^{-1}.$$

Hence, a choice of v_1, v_2 is

$$v_1(x) = -\frac{1}{25} x^5, \qquad v_2(x) = \frac{1}{5} \ln |x|.$$

A particular solution of the non-homogeneous equation is thus

$$y_p = v_1 y_1 + v_2 y_2 = x^2 (-\frac{1}{25} + \frac{1}{5} \ln |x|).$$

Therefore, the general solution of the equation is

$$c_1 x^{-3} + c_2 x^2 + \frac{1}{5} x^2 \ln |x|,$$

for arbitrary constants c_1, c_2.

7. Suppose that Bob graduates with \$1,000,000 credit card debt. Assume that the card charges 1% annual interest, compounding continuously. Also assume that Bob pays \$1,000 per year, continuously. Find out how long it will take for Bob to pay the debt.

Solution. Let $S(t)$ be the amount Bob owes at time t and assume the current time is zero. Thus, $S(0) = 1,000,000$. Our goal is to find the value of t such that $S(t) = 0$. By the information given, the equation that $S(t)$ satisfies is

$$S' = 0.01S - 1000.$$

Of course, we can solve this equation by noting either it is linear or separable. There is yet another way to solve this particular kind of equations, called the z-substitution, which we use here. Let $z = S' = 0.01S - 1000$, then the equation that z satisfies is

$$z' = (0.01S - 1000)' = 0.01S' = 0.01z.$$

Therefore,

$$z(t) = Ce^{0.01t},$$

and

$$S(t) = \frac{z(t) + 1000}{0.01} = \frac{Ce^{0.01t} + 1000}{0.01} = De^{0.01t} + 100,000$$

for some constant D.

Now use the initial value $S(0) = 1,000,000$, and we have $D = 9,900,000$. We realize that $S(t)$ is increasing in t, since $D > 0$, which tells us that it is impossible for the poor Bob to ever finish paying the debt with a paying rate of \$1,000 per year.