LECTURE 6: VARIATION OF PARAMETERS

1. Variation of Parameters

1.1. **Worksheet for Variation of Parameters.** In this section, we are going to learn a method which can be used to find particular solutions for non-homogeneous linear second order ODEs, called *the variation of parameters*. The prerequisite of using this method is that we know a fundamental set of solutions of the corresponding homogeneous equation. And the idea is to see if we could obtain a particular solution of the non-homogeneous equation by combining the solutions to the homogeneous equations. Here, we will work through an example and see how this idea works. The equation for which we are going to find a particular solution is

$$y'' + 2y' + y = \cos x.$$

(1) First, let us not forget that the method of undetermined coefficients applies to this equation. Find a particular solution taking this approach.

Answer. Here we use the complex trick. Let $z(x)$ be a complex valued function. It follows that if z satisfies the equation

$$z'' + 2z' + z = e^{ix},$$

then the real part of z, $\mathrm{Re}(z)$, satisfies the original equation. Since i is not a root of the characteristic polynomial $p(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$, our guess of the solution z is

$$z(x) = Ae^{ix},$$

where A is some undetermined complex constant. Plugging this form in the equation, we have

$$(-A + 2iA + A)e^{ix} = e^{ix}.$$

Therefore, $A = -\frac{i}{2}$ and

$$y(x) = \mathrm{Re}(z(x)) = \mathrm{Re}\Big(-\frac{1}{2}ie^{ix}\Big) = \mathrm{Re}\Big(-\frac{1}{2}(i\cos x - \sin x)\Big) = \frac{1}{2}\sin x,$$

is a particular solution to the equation $y'' + 2y' + y = \cos x$.

(2) The method of variation of parameters proposes another way to solve the original equation. As mentioned, let us first find a fundamental set of solutions of the homogeneous equation

$$y'' + 2y' + y = 0,$$

and denote it as $\{y_1, y_2\}$.

Answer. Note that this equation is constant coefficient and the characteristic polynomial is $p(\lambda) = (\lambda + 1)^2$. We know by the theory of constant coefficient homogeneous linear equations that a fundamental set of solutions is $\{e^{-x}, xe^{-x}\}$.

(3) Now supposing that $u_1(x)y_1(x) + u_2(x)y_2(x)$ solves the equation $y'' + 2y' + y = \cos x$ for some functions $u_1(x)$ and $u_2(x)$, what equations should u_1, u_2 satisfy? Can we find such functions u_1, u_2?

Date: 05/20/15.

Answer. Denoting $u_1(x)y_1(x) + u_2(x)y_2(x)$ as $z(x)$, we need that

$$z'' + 2z' + z = \cos x.$$

This is

$$\begin{aligned}
\cos x &= (u_1y_1 + u_2y_2)'' + 2(u_1y_1 + u_2y_2)' + (u_1y_1 + u_2y_2) \\
&= (u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2')' + 2(u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2') + (u_1y_1 + u_2y_2) \\
&= (u_1'y_1 + u_2'y_2)' + (u_1'y_1' + u_2'y_2') + (u_1y_1'' + u_2y_2'') \\
&\quad + 2(u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2') + (u_1y_1 + u_2y_2) \\
&= (u_1'y_1 + u_2'y_2)' + 2(u_1'y_1 + u_2'y_2) + (u_1'y_1' + u_2'y_2') + u_1(y_1'' + 2y_1' + y_1) \\
&\quad + u_2(y_2'' + 2y_2' + y_2) \\
&= (u_1'y_1 + u_2'y_2)' + 2(u_1'y_1 + u_2'y_2) + (u_1'y_1' + u_2'y_2'),
\end{aligned}$$

where we have used the fact that y_1, y_2 are solutions to the homogeneous equation to eliminate the terms $y_1'' + 2y_1' + y_1$ and $y_2'' + 2y_2' + y_2$. One sufficient condition for the equality above to hold is

$$u_1'y_1 + u_2'y_2 = 0, \qquad u_1'y_1' + u_2'y_2' = \cos x.$$

These two equations are perhaps more recognizable if we put them in the matrix form

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \cos x \end{pmatrix}.$$

From linear algebra, we know that in the domain where the equation has existence and uniqueness property, if y_1 and y_2 form a set of fundamental solutions, then the Wronskian

$$W(y_1, y_2) = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

is nonzero for all values of x. Therefore, the coefficient matrix is always invertible and we have

$$\begin{aligned}
\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \cos x \end{pmatrix} \\
&= \frac{1}{W(y_1, y_2)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ \cos x \end{pmatrix}.
\end{aligned}$$

Up to this point, our calculation has been symbolic. Now let us plug in $y_1 = e^{-x}$ and $y_2 = xe^{-x}$. The result is

$$\begin{aligned}
u_1' &= -x \cos x e^x, \\
u_2' &= \cos x e^x.
\end{aligned}$$

Taking the antiderivatives would give us the choices

$$\begin{aligned}
u_1 &= -\frac{1}{2}(x \cos x e^x + x \sin x e^x - \sin x e^x), \\
u_2 &= \frac{1}{2}(\sin x e^x + \cos x e^x).
\end{aligned}$$

Finally,

$$\begin{aligned} u_1y_1 + u_2y_2 &= -\frac{1}{2}(x\cos xe^x + x\sin xe^x - \sin xe^x)e^{-x} + \frac{1}{2}(\sin xe^x + \cos xe^x)xe^{-x} \\ &= \frac{1}{2}\sin x, \end{aligned}$$

which coincides with the particular solution we obtained using the undetermined coefficients.

1.2. **Remarks.** (1) As you may have noticed, the thought behind the variation of parameters resembles those behind the method of integrating factors or reduction of order, in that one seem to be asking: can we multiply the solution or the equation by a function so that the equation is simplified or other solutions can be obtained from doing this. One technical difference is that in establishing the equations for u_1, u_2 in the current problem, we did not go straight forwardly to the end and write down expressions involving u_1'' and u_2'', but instead, we picked out the expressions $u_1'y_1 + u_2'y_2$ and $u_1'y_1' + u_2'y_2'$ in the middle of the calculation and observed that there is an obvious way to set values to them. This makes it a lot easier to figure out what u_1 and u_2 are in the end.

(2) We have seen that the method of variation of parameters is capable of solving non-homogeneous constant coefficient equations which can be solved by undetermined coefficients, but in a more complex way.

(3) The method of variation of parameters applies to not only constant coefficient non-homogeneous equations, but to second order linear equations in general, provided that fundamental solutions of the corresponding homogeneous equation are given. This part will be summarized in the next section. In fact, variation of parameters applies to higher order linear ODEs, but we will not go into the details here.

2. Summary

2.1. **The Method of Variation of Parameters.** Here we summarize the main result of variation of parameters for second order non-homogeneous linear ODEs in general. Consider the equation

$$y'' + p(x)y' + q(x)y = g(x).$$

Let y_1, y_2 form a fundamental set of solutions to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$. We could always choose functions u_1, u_2 such that $u_1y_1 + u_2y_2$ is a particular solution of the original equation, where u_1, u_2 can be computed by

$$\begin{pmatrix} u_1'(x) \\ u_2'(x) \end{pmatrix} = \frac{1}{W(y_1, y_2)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

Sketch of Proof. Assuming that $z = u_1y_1 + u_2y_2$ is a solution of the original equation, then, plugging this in the equation, we could obtain

$$g(x) = (u_1'y_1 + u_2'y_2)' + p(x)(u_1'y_1 + u_2'y_2) + (u_1'y_1' + u_2'y_2').$$

So an obvious choice of u_1, u_2 is such that

$$u_1'y_1 + u_2'y_2 = 0, \qquad u_1'y_1' + u_2'y_2' = g(x).$$

Written in the matrix form, this is just

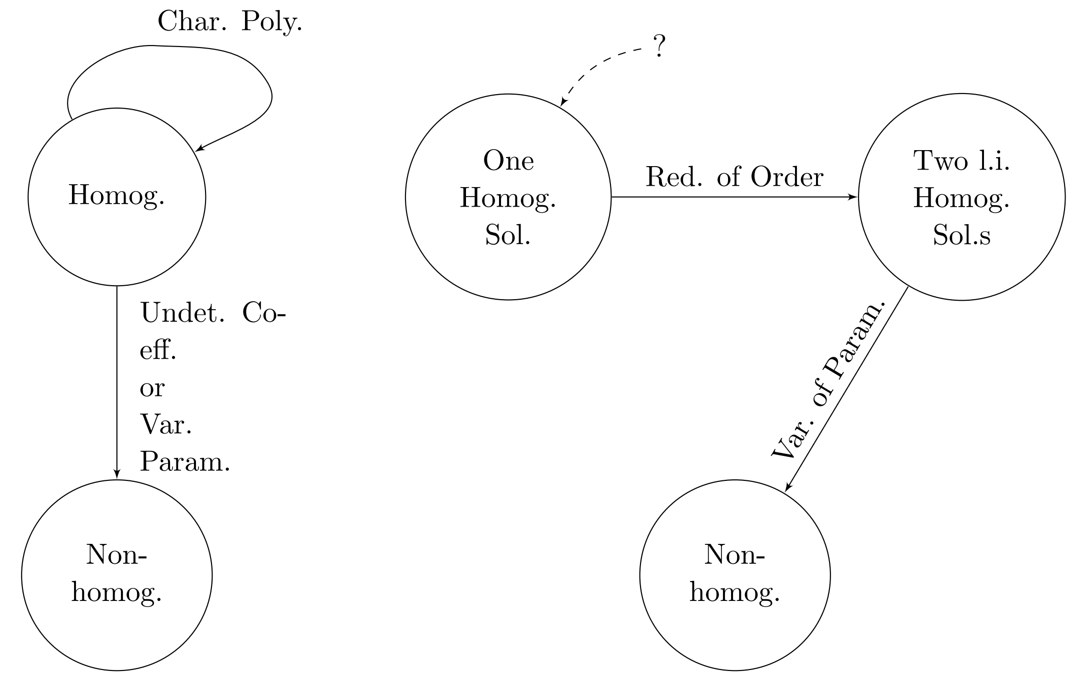
$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

Noting that y_1, y_2 are linearly independent, our conclusion follows.

2.2. **Second order linear ODEs: Summary.** Until now, we have considered second order linear ordinary differential equations in the following aspects:

- Constant coefficients or not?
- Homogeneous or not?
- In the homogeneous case, what if we know one solution already?
- In the non-homogeneous case, what if a fundamental set of solutions for the homogeneous equation is known?

The corresponding results can be put in a chart:



(A) Constant Coefficients (B) Non-constant Coefficients

Figure 1. Second Order Linear ODEs

3. Appendix: The anti-derivative of $xe^x \cos x$?

Here we present a way to find the anti-derivative of $xe^x \cos x$. First, we would guess that the anti-derivative would involve terms such as $xe^x \cos x, xe^x \sin x$ and maybe $e^x \cos x, e^x \sin x$. Let us calculate the derivatives of these functions.

$$(xe^x \cos x)' = e^x \cos x + xe^x \cos x - xe^x \sin x, \tag{1}$$

$$(xe^x \sin x)' = e^x \sin x + xe^x \sin x + xe^x \cos x, \tag{2}$$

$$(e^x \cos x)' = e^x \cos x - e^x \sin x, \tag{3}$$

$$(e^x \sin x)' = e^x \sin x + e^x \cos x. \tag{4}$$

Our goal is to combine these equations so that the right hand side reduces to a multiple of $xe^x \cos x$. It can be seen that adding the first two equations partially does the job, with two extra terms $e^x \cos x + e^x \sin x$. One realizes immediately that these two extra terms can be replaced using the last equality above. So we have, calculating $(1) + (2) - (4)$:

$$(xe^x \cos x + xe^x \sin x - e^x \sin x)' = 2xe^x \cos x.$$

Therefore, the anti-derivative of $xe^x \cos x$ is

$$\frac{1}{2}(xe^x \cos x + xe^x \sin x - e^x \sin x).$$

As an exercise, try to find the anti-derivative of $xe^x \sin x$.