LECTURE 5: SECOND ORDER LINEAR ODES

1. Introduction

Hopefully, this lecture will be mostly review for each of us. We discuss the second order linear ordinary differential equations. To start with, by the definition of linearity and order of ODEs, we know that these equations in general take the form:

$$P(x)y'' + Q(x)y' + R(x)y = S(x),$$

for some continuous functions $P(x), Q(x), R(x), S(x)$. Again, to make the situation easier to deal with, we assume that $P(x)$ is always nonzero in the domain of definition, so the form of the equations can be simplified to

$$y'' + p(x)y' + q(x)y = g(x).$$

Remember, when $g(x) \equiv 0$, this equation is called *homogeneous*. Lying at the foundation is perhaps the theorem of existence and uniqueness of solutions:

Theorem (Existence & Uniqueness) For the initial value problem

$$y'' + p(x)y' + q(x)y = g(x), \qquad y(t_0) = y_0, y'(t_0) = d_0,$$

if $p(x), q(x), g(x)$ are continuous within some interval I containing t_0, then solution exists and is unique on the interval I.

The power of this theorem can be seen when we take into account the *linearity* of the equations. Remember that linearity implies that all the solutions of a homogeneous equation form a vector space, that is, the sum and scalar multiple of solutions remain solutions. Moreover, the mapping

$$\phi : y(x) \mapsto (y(t_0), y'(t_0))^T$$

is a one-to-one and onto (by the Theorem) linear transformation between the solution space and $\mathbb{R}^2$. It follows that the solution space is real two dimensional. In other words, for linear homogeneous ODEs of second order, once two linearly independent solutions are found, then all solutions are linear combinations of these two fundamental solutions, and the coefficients can be determined by the initial values.

Another use of linearity is in the argument that in order to find all the solutions of a non-homogeneous equation, it suffices to find *a particular solution* and the *the fundamental set of solutions to the corresponding homogeneous equation*. We have seen this before. Certain results will be summarized in the last section.

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2. Constant Coefficient Homogeneous Linear ODEs, Complex Roots

For constant coefficient homogeneous linear ODEs, a standard model is the harmonic oscillator (without external force),

$$m\ddot{x} + f\dot{x} + kx = 0,$$

where the constant coefficients are positive. In general, we could allow negative coefficients. For solving these equations, a common approach is to test whether there are solutions in the form

$$x(t) = e^{rt}.$$

Plugging this in the equation gives,

$$(mr^2 + fr + k)e^{rt} = 0.$$

Now, whenever r is a root of the polynomial $mr^2 + fr + k$ (called the *characteristic polynomial*), $x = e^{rt}$ is a solution of the original equation. By the quadratic formula, the roots are

$$r = \frac{-f \pm (f^2 - 4mk)^{1/2}}{2}.$$

There are three cases:

(1) $f^2 - 4mk > 0$. There exist two distinct real roots r_1, r_2. One can show that in this case $e^{r_1 t}, e^{r_2 t}$ are linearly independent, hence forming a fundamental set of solutions.

(2) $f^2 - 4mk < 0$. There exist two distinct complex roots $a + ib, a - ib$. In one of the homework exercises, you will be asked to show when r is complex, we still have $\frac{d}{dt}e^{rt} = re^{rt}$. So formally, $e^{(a+ib)t}$ and $e^{(a-ib)t}$ both satisfy the equation, but this is essentially

$$\begin{aligned}
&m(e^{at}(\cos bt + i\sin bt))'' + f(e^{at}(\cos bt + i\sin bt))' + k(e^{at}(\cos bt + i\sin bt)) \\
=&[m(e^{at}\cos bt)'' + f(e^{at}\cos bt)' + k(e^{at}\cos bt)] + i[m(e^{at}\sin bt)'' + f(e^{at}\sin bt)' + k(e^{at}\sin bt)] \\
=&0.
\end{aligned}$$

By looking at the real and imaginary parts alone, we obtain that $e^{at}\cos bt$ and $e^{at}\sin bt$ are solutions to the equation. One can use the Wronskian to check that they are linearly independent. Hence, $\{e^{at}\cos bt, e^{at}\sin bt\}$ is a fundamental set of solutions.

(3) $f^2 - 4mk = 0$. In this case, we have a repeated root r and we learned that the fundamental solutions are e^{rt}, te^{rt}. See the next section for a general technique which gives us the reason behind this result.

3. Repeated Roots, Reduction of Order

In the realm of second order homogeneous linear ODEs, one particularly useful technique in finding solutions is called *the reduction of order*. The motivation behind this idea might be the attempt to answer the question: supposing that one solution $y(x)$ of the equation $y'' + p(x)y' + q(x)y = 0$ is known, can we find another solution which is in the form $v(x)y(x)$ for some non-constant function $v(x)$? (Sounds familiar? Yes, similar thought is used in deriving the integrating factors for the first order linear equations.)

Now suppose that $y(x)$ is known for solving the equation $y'' + p(x)y' + q(x)y = 0$. Let $v(x)y(x)$ be another solution. Then we have:

$$\begin{aligned}
&(vy)'' + p(vy)' + q(vy) \\
=&(v'y + vy')' + p(v'y + vy') + q(vy) \\
=&(v''y + 2v'y' + vy'') + p(v'y + vy') + q(vy) \\
=&v''y + (2y' + py)v' + v(y'' + py' + qy) \\
=&v''y + (2y' + py)v' \\
=&0.
\end{aligned}$$

Thus, $z = v'(x)$ satisfies the first order separable equation:

$$z'y + (2y' + py)z = 0.$$

Thus, z can be solved. Finally, $v(x)$ can be found by a direct integration of $z(x)$.

As an example, we use the reduction of order to complete the discussion in the case of repeated roots of the constant coefficient homogeneous equations. Recall that a solution is known to be $y(t) = e^{rt}$, where $r = -\frac{1}{2m}f$, because $-f/m$ is the sum of the (repeated) roots. Hence the function $v(t)$ satisfies

$$0 = v''e^{rt} + (2(e^{rt})' + \tfrac{f}{m}e^{rt})v' = v''e^{rt} + (2r + \tfrac{f}{m})e^{rt}v' = v''e^{rt}.$$

This is,

$$v'' = 0.$$

One choice of $v(t)$ which is not constant is of course $v(t) = t$, as desired.

4. Non-homogeneous Equations, Undetermined Coefficients

This section aims at finding particular solutions for constant coefficient non-homogeneous second order linear ODEs. Again, this should be a review and the main theorem is the following.

Theorem. Consider the constant coefficient linear equation

$$ay'' + by' + cy = g(x),$$

with the characteristic polynomial $p(\lambda) = a\lambda^2 + b\lambda + c$.

(1) If $g(x) = P_n(x)$ is a polynomial of degree n, then $y_p = x^s(a_n x^n + ... + a_1 x + a_0)$, where s is the multiplicity of 0 as a root of $p(\lambda)$.

(2) If $g(x) = P_n(x)e^{rx}$, then $y_p = x^s(a_n x^n + ... + a_1 x + a_0)e^{rt}$, where s is the multiplicity of r as a root of $p(\lambda)$.

(3) If $g(x) = P_n(x)e^{ax}\cos bx$ or $g(x) = P_n(x)e^{ax}\sin bx$, then $y_p = x^s[(a_n x^n + ... + a_1 x + a_0)e^{at}\cos bx + (b_n x^n + ... + b_1 x + b_0)e^{at}\sin bx]$, where s is the multiplicity of $a + ib$ as a root of $p(\lambda)$.

Remarks. (1) You may wonder, why could we always assume solution to take certain form when $g(x)$ is in one of the three cases above? The reason is, we would expect some function, after taking whose derivatives we'd obtain expressions in the form of $g(x)$ and that the coefficients in the guesses above can always be solved for. For more details, refer to the proof of this theorem in the textbook.

(2) Consider the equation

$$y'' + p(x)y' + q(x) = x^2 + \cos x.$$

Now $g(x) = x^2 + \cos x$ which is not in one of the three forms described in the theorem. Here is another place where the linearity of the equation becomes helpful, as one could find solutions for

$$y'' + p(x)y' + q(x) = x^2,$$

and

$$y'' + p(x)y' + q(x) = \cos x,$$

separately using the theorem and call them $y_1(x)$ and $y_2(x)$. Then, $y_1(x) + y_2(x)$ solves the original equation.