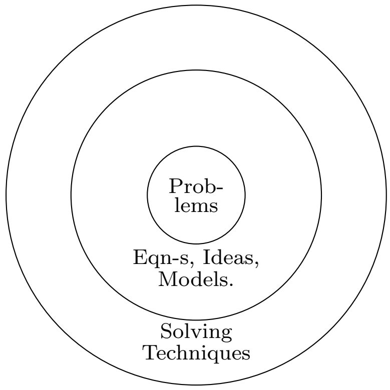
LECTURE 4: EXACT EQUATIONS AND INTEGRATING FACTORS, EULER'S METHOD

1. A Very General Remark

Before starting our discussion today, let us remind ourselves what we have learned so far. We have defined what differential equations are, had a rough classification of these equations, and learned how to solve the simplest of these classes, the first order linear ODEs, by a method called integrating factors. We then challenged ourselves with first order nonlinear ODEs, and discovered that a certain type of equations, called separable equations, can be solved by separating the variables then integrate. We looked at both population models and physics models, and introduced the vector field plot to help us visualize the solutions of these equations with various initial values. For autonomous equations, a more convenient way of qualitative analysis is the phase line, with which we figured out the stability of the equilibria of all the population models we have previously learned.



In a nutshell, we had certain real-life problems or observations or curiosity (at the core), we had mathematical models or equations as approaches (in the mid-layer), and we had techniques that are used to analyze or solve the equations and models (in the outer layer). Usually, we start from the outer layer, introduce the techniques and use them to solve problems. However, one should remember that it is the core problems that drive ideas thus inspire new solutions. In the future, we will see in the outer layer various methods such as the Laplace transform, the power series approach, the Fourier transform, various numerical methods, etc. At the "idea" level, there are the established wave equations, heat equations, the idea of decomposing a function and solving for each component individually, the idea of approximation, and so on. At the core, finally, there is the wish to understand how population evolves over time, how water waves propagate, how heat transfers in materials, how to predict weather and why it is hard to predict, how do planets rotate around the sun, how do the existing matters affect the shape of our universe, and so on. The take-away?

2. Exact Equations and Integrating Factors

In this section, we are going to learn to solve a type of first order ODEs which can be put in the following form:

$$M(x,y) + N(x,y)y' = 0.$$

Instead of explaining the techniques directly, we will walk through a series of problems, and "discover" the general pattern by ourselves.

Date: 05/18/15.

2.1. **Worksheet for Exact Equations.**
(1) Let $F(x, y)$ be a differentiable function in x, y. Assume that $y(x)$ is a differentiable function such that $F(x, y(x))$ is defined. Write down the chain-rule formula which computes the derivative

$$\frac{dF(x, y(x))}{dx}.$$

Answer.

$$\frac{dF(x, y(x))}{dx} = \frac{\partial F}{\partial x}(x, y(x)) + \frac{\partial F}{\partial y}(x, y(x))\frac{dy}{dx}(x).$$

(2) Now we want to solve the first order non-linear, non-separable ODE:

$$2x + y^3 + 3xy^2\frac{dy}{dx} = 0.$$

First, apply your answer in part (1) to calculate the following derivative, viewing y as a function in x,

$$\frac{d(xy^3 + x^2)}{dx}.$$

What do you notice? What is the solution to the equation above?

Answer. We have

$$\frac{d(xy^3 + x^2)}{dx} = y^3 + 2x + 3xy^2\frac{dy}{dx},$$

which is exactly the left hand side of the equation. Hence the solution to the equation is simply such $y(x)$ that satisfies:

$$xy^3 + x^2 = C,$$

for some constant C.

(3) If you are now thinking of a generalization, you may notice, we can put the above equation in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0,$$

and in particular, there exists a function $F(x, y)$ such that

$$M(x, y) = \frac{\partial F}{\partial x}, \qquad N(x, y) = \frac{\partial F}{\partial y}. \qquad (*)$$

Convince yourself that, in general, if such an F exists, the solutions to the equation are implicitly given by

$$F(x, y) = C.$$

(4) Show that if there exists an F so that the conditions in $(*)$ hold, then $M(x, y)$ and $N(x, y)$ must satisfy the following:

$$\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y).$$

Answer. We have

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N(x, y)}{\partial x}.$$

(5) Now we know that the condition in part (4) is necessary for the function $F(x, y)$ to exist. In this part, we show that this condition is also sufficient.

- Given that $\frac{\partial F}{\partial x} = M(x,y)$, what form should $F(x,y)$ take? In your expression of F, what is still unknown? Hint: The Fundamental Theorem of Calculus.
  Answer.
  $$F(x,y) = \int_{x_0}^{x} M(t,y)dt + h(y),$$
  where $h(y)$ is an unknown function.
- Given that $\frac{\partial F}{\partial y} = N(x,y)$, can we fix the form of $F(x,y)$?
  Answer. We need
  $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\int_{x_0}^{x} M(t,y)dt + \frac{d}{dy}h(y) = N(x,y).$$
  Thus, $h(y)$ satisfies
  $$\frac{dh(y)}{dy} = N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(t,y)dt.$$
  OK, let's say
  $$h(y) = \int \Big(N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(t,y)dt\Big)dy? \qquad (**)$$
- What's wrong in the previous answer?
  Answer. The problem is that, in order to obtain $h(y)$ by the integral $(**)$, we need the integrand to be independent on x. To show this, take the derivative of the integrand with respect to x, and get
  $$\begin{aligned}&\frac{\partial}{\partial x}\Big(N(x,y) - \frac{\partial}{\partial y}\int_{x_0}^{x} M(t,y)dt\Big)\\ &= \frac{\partial}{\partial x}N(x,y) - \frac{\partial^2}{\partial x \partial y}\int_{x_0}^{x} M(t,y)dt\\ &= \frac{\partial}{\partial x}N(x,y) - \frac{\partial^2}{\partial y \partial x}\int_{x_0}^{x} M(t,y)dt\\ &= \frac{\partial}{\partial x}N(x,y) - \frac{\partial}{\partial y}M(x,y)dt\\ &= 0.\end{aligned}$$
  Hence the integrand in $(**)$ is independent on x, and the problem is fixed.

(6) Use the method we've developed to solve the equation
$$(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)\frac{dy}{dx} = 0.$$

[Example 2, p.98]

2.2. **Remark.** Equation
$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$
is said to be *exact* if there exists a function $F(x,y)$ such that $\frac{\partial}{\partial x}F(x,y) = M(x,y)$ and $\frac{\partial}{\partial y}F(x,y) = N(x,y)$, that is, the equation can be rewritten as
$$\frac{d}{dx}F(x,y(x)) = 0.$$
We have shown that this equation is exact if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

2.3. **Worksheet for Integrating Factors.**
(1) Indeed, not all equations of the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ are exact. If there exists a function $\mu(x,y)$, so that the equation

$$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{dy}{dx} = 0$$

is exact. What equation should $\mu(x,y)$ satisfy?
Answer. We need

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N).$$

This is just

$$M\frac{\partial \mu}{\partial y} - N\frac{\partial \mu}{\partial x} + \Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big)\mu = 0.$$

(2) Note that the equation that μ satisfies may still be hard to solve. But what if we guess wildly that $\mu(x,y)$ only depends on x? What equation should μ satisfy then? What relation should $M(x,y), N(x,y)$ satisfy? Do the same analysis for the guess that $\mu(x,y)$ only depends on y.
Answer. For $\mu = \mu(x)$, the equation about μ simplifies to

$$-N(x,y)\frac{d\mu}{dx} + \Big(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\Big)\mu(x) = 0$$

Of course, this equation is not solvable unless $\frac{M_y - N_x}{N}$ is a function of x. If this is satisfied, $\mu(x)$ can be found by direct integration.

(3) Following the discussion in part (2), solve the differential equation

$$(3xy + y^2) + (x^2 + xy)y' = 0,$$

assuming that we know that there exists an integrating factor $\mu(x)$. [Example 4, p.100]

3. Euler's Method

In the experiment below, we are going to explore a numerical method for solving first order ODEs, called Euler's Method. The idea is simple: use continuous line segments to approximate solution curves. To be more precise, supposing that (x_0, y_0) is on the solution curve, then for small Δx, we could use $y_0 + \Delta x \cdot f(x_0, y_0)$ as an approximation of $y(x_0 + \Delta t)$. Euler's method suggests us continuing this process, taking $(x_1, y_1) = (x_0 + \Delta x, y_0 + \Delta x \cdot f(x_0, y_0))$ as the new starting point, using the accurate tangent slope at this point, and carry out the approximation to obtain $(x_2, y_2) = (x_1 + \Delta x, y_1 + \Delta x \cdot f(x_1, y_1))$, and so on. A pseudocode for this method is as follows.

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procedure Euler's Method for Equation $\frac{dy}{dx} = f(x,y)$
  $(x_0, y_0) \leftarrow initialValue$
  $\Delta x \leftarrow stepSize$
  $N \leftarrow numSteps$
loop:
  for $(i = 0, i < N, i++)$ **do**
    $x_{i+1} = x_i + \Delta x;$
    $y_{i+1} = y_i + f(x_i, y_i)\Delta x;$

For the initial value problem

$$\frac{dy}{dx} = 1 - x - 0.5y, \qquad y(0) = 1,$$

we use Mathematica® to carry out the Euler's method for step sizes $\Delta x = 1, 0.5, 0.1$ to approximate the solution within $\{0 \le x \le 5\}$. The code and the result are shown below, where the orange and blue curves represent accurate solutions and approximate solutions, respectively.

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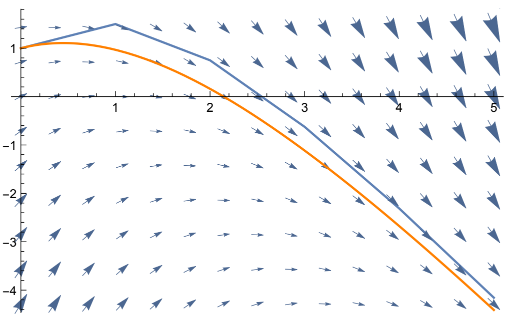
procedure MATHEMATICA® FOR $\frac{dy}{dx} = 1 - x - 0.5y, y(0) = 1$

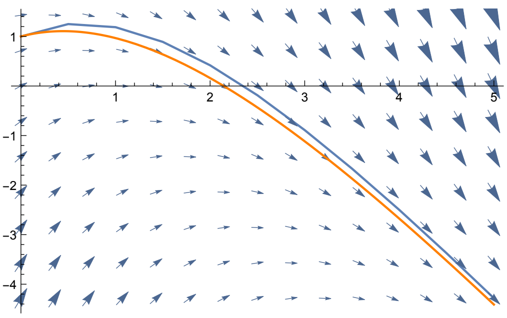
Clear[y];
y[0]=1;
stepSize=1; # set to be 0.5, 0.1 later on
Do[y[n+1]=y[n]+stepSize*(1-stepSize*n-0.5*y[n]),{n,0,5/stepSize-1}];
y[5/stepSize];

table=Table[{N[stepSize*k],y[k]},{k,0,5/stepSize}]
a=ListPlot[table, Joined→True]; #Plotting the piecewise-linear approximation
b=VectorPlot[{1,1-x-0.5*y},{x,0,5},{y,-5,5}]; #Plotting the vector field
c=Plot[{$-2e^{-0.5x}(2.5-3e^{0.5x}+e^{0.5x}x)$},{x,0,5}, PlotStyle→Orange]; #Plotting the accurate solution

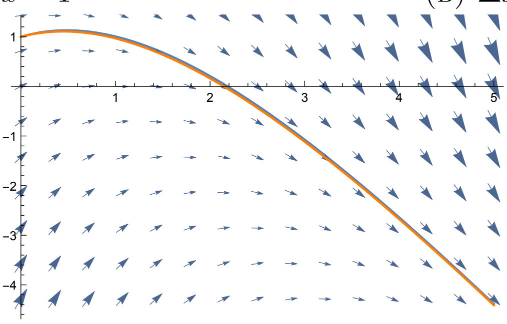
Show[a,b,c]



(A) $\Delta x = 1$



(B) $\Delta x = 0.5$



(C) $\Delta x = 0.1$

FIGURE 1. $\frac{dy}{dx} = 1 - x - 0.5y, \qquad y(0) = 1$

Next, for the initial value problem slightly modified from the one above (with only a sign change),

$$\frac{dy}{dx} = 1 - x + 0.5y, \qquad y(0) = 1,$$

we have

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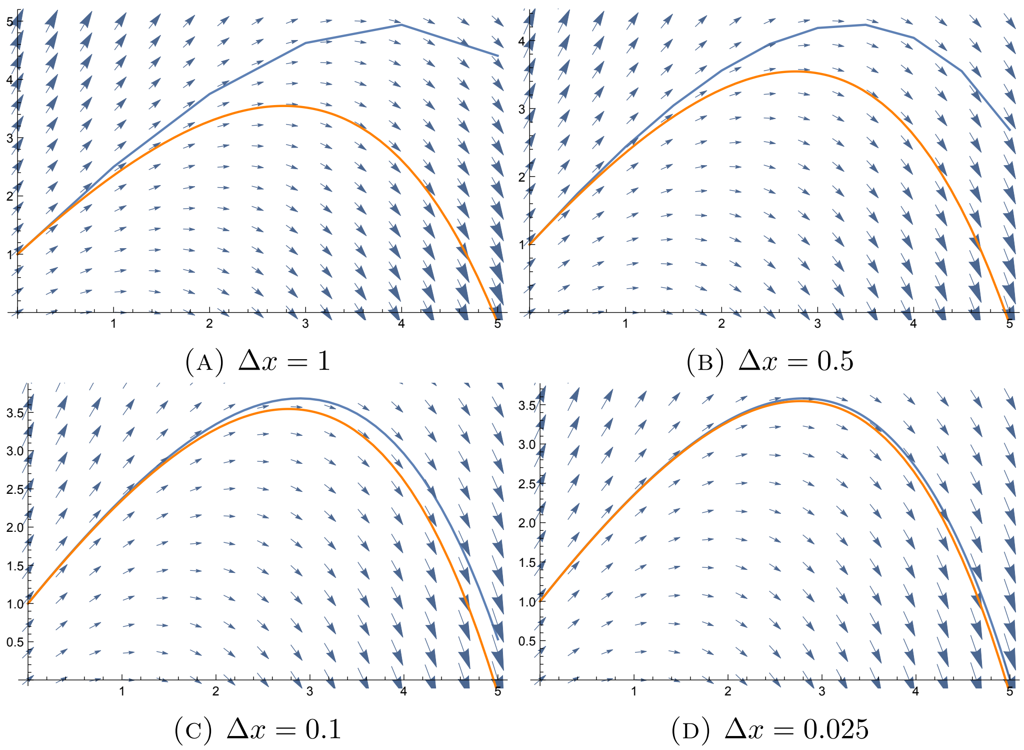
procedure MATHEMATICA® FOR $\frac{dy}{dx} = 1 - x + 0.5y, y(0) = 1$

Clear[y];
y[0]=1;
stepSize=1; # set to be 0.5, 0.1,0.025 later on
Do[y[n+1]=y[n]+stepSize*(1-stepSize*n+0.5*y[n]),{n,0,5/stepSize-1}];
y[5/stepSize];

table=Table[{N[stepSize*k],y[k]},{k,0,5/stepSize}]
a=ListPlot[table, Joined→True]; #Plotting the piecewise-linear approximation
b=VectorPlot[{1,1-x+0.5*y},{x,0,5},{y,-5,5}]; #Plotting the vector field
c=Plot[{$2 - e^{0.5x} + 2x$},{x,0,5}, PlotStyle→Orange]; #Plotting the accurate solution

Show[a,b,c]



(A) $\Delta x = 1$

(B) $\Delta x = 0.5$

(C) $\Delta x = 0.1$

(D) $\Delta x = 0.025$

FIGURE 2. $\frac{dy}{dx} = 1 - x + 0.5y, \qquad y(0) = 1$

By looking at the figures, you may have, but are not restricted to, the following four observations:

(1) The piecewise-linear approximations have better accuracy when we choose smaller step sizes.

(2) Within each line segment of the piecewise-linear approximation, the slope at the starting point is always accurate, even though the starting point itself may not be accurate.

(3) Euler's method works better for the first equation than the second.

(4) The vector field for the first equation appears "converging" while that for the latter appears "diverging".

For the first observation, you may want a rigorous proof. Here, I recommend you solve through Ex.20, Sec. 2.7. For the second observation, we have more to say. The accuracy of the slope is just how we define the Euler's method. The inaccuracy of the end points are exactly the reason why error might accumulate during the procedure. Combining with the fourth observation, we see that when the vector field diverges, one has to pay more for the inaccuracy occurred the middle, and the error at the end could be large, unless we choose Δx to be really, really small (Figure 2).