LECTURE 3: MODELING WITH FIRST ORDER EQUATIONS, AUTONOMOUS EQUATIONS

1. Modeling with First Order Equations

In this lecture, we put the first order ODEs in real-life use: modeling. First let's not forget our old friend: the population model without environmental constraint

$$\dot{p} = rp,$$

which represents the ideal situation in which the reproduction rate of the population at any given time is proportional (by a constant r) to the size of the population at that moment. Technically, we know pretty well how to solve this equation, by observing either that it is linear or that it is separable.

To generalize a bit, looking at the model itself, you may notice that it is in fact capable of modeling any situation in which the unknown quantity augments at a rate proportional to the amount of this quantity at any time. Hopefully this reminds you of the money that you've put in the bank a while ago or some kind of investment. If we let $S(t)$ to be the value of your initial investment $S(0)$ at time t and suppose that the interest compounds continuously at a constant rate, we will have the equation

$$\dot{S} = rS,$$

which looks exactly the same as the population model.

Continuing with the investment model, but now supposing that you decide to compound a part of your income continuously into the investment. Or, if you like the population model better, suppose that there is some continuous incoming population at a certain rate. How, then, should we modify our models? Again, we set our unknown function to be $S(t)$ in keeping with the investment model. Now the rate of change in $S(t)$ consists of two parts: the rate of increase generated by the bank interest and the continuous compounding of the investment. If we assume that the investment is compounded at a constant rate k, then we have the equation:

$$\dot{S} = rS + k.$$

Again, this equation is both linear and separable, and we know how to solve it.

Caution: In establishing equations like these, one needs to be careful about the units. For instance, if r is given as a number per day while k is a number per year, then we will need to scale these parameters so that they are in the same unit.

In the examples above, we know immediately what the unknown function is just by looking at the situation itself. However, sometimes, figuring out the correct unknown function requires a bit of thought, and this is crucial in setting up models. For example, this time, instead of saving money, one borrows a certain amount of money at the

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beginning and start your payment immediately and continuously at a certain constant rate k. Suppose that the loan has a constant interest r compounding continuously. How can we set a model to figure out the time it takes to finish the payment? The question asks for a time, but how do we characterize the time? If we think of the time as a moment when certain amount of payment is made, this will not work well since the payed amount itself depends on t. But what about the amount owed? If we set $W(t)$ to be the amount that is still owing, then the finishing time t_0 can be characterized simply as $W(t_0) = 0$. Moreover, we know about the rate of change of $W(t)$: at any moment, it increases at the rate $rW(t)$ by the interest of the loan, and it decreases at the rate k because of the payment. So, the model is clear:

$$\dot{W} = rW - k.$$

Leaking Tank. Now, let us turn to a problem in physics: Suppose that a tank is given with height H and areas of cross sections $A(h)$, which is initially filled with water. Now suppose that there is a hole of cross area a at the bottom of the tank. How long does it take for the water to drain down to the bottom of the tank?

Again, our first question is, how to set the unknown function? Hopefully, your answer is: the height of the water level relative to the bottom of the tank at any time t. Let this quantity be denote as $h(t)$. Of course, $h(t)$ changes, but at what rate? It is time to think of the *relations* involved in the system that we are considering. Water level changes because there is a loss of water. At any time t, the rate of water loss is thus related to the decrease rate in the height of the water level and equals $A(h(t))\dot{h}$. On the other hand, the rate of water loss can be characterized by the area of the outlet a times the instant velocity v of the outgoing water.

Now it is time to use some physics. If we ideally assume that the system under consideration has conserved energy, then what happens is that a ΔV-small volume of water at the surface level instantly shifts to the level of the outlet. The loss of the potential energy must be equal to the gain of the kinetic energy. Thus we have

$$\rho g \Delta V h = \frac{1}{2}\rho \Delta V v^2.$$

Hence $v = (2gh)^{1/2}$.

Finally, combining what we've obtained so far, we have the equation:

$$A(h)\dot{h} = -a(2gh)^{1/2}.$$

Note that this equation is separable. For $A(h)$ equals some constant, you can try to solve this equation as an exercise.

***Tautochrone curve.** If we fix a point p on a circle and roll the circle without slipping along a line, then the curve traced out by p is called a *cycloid curve*. A famous problem in history is: given two points of different height in space, find the curve of fastest descent from one point to the other, without friction. It turns out that this problem can be reduced to solving a first order ordinary differential equation. The reduction itself could be by way of a more advanced method called *the calculus of variations*. And the solution, surprisingly, coincides with sections of the (upside-down) cycloid curves. Another nice property which is fun to imagine about is that, from being static, it takes the same amount of time to descend along a cycloid curve to its lowest point, wherever we set the initial position. This is why such curves are named tautochrone, which translates into "same time" in English.

For those who are interested, try solving Ex. 32 in section 2.3. More historical remarks are available on Wikipedia.

2. Autonomous Equations and Population Dynamics

Recall once more the simplest population model:

$$\dot{p} = rp,$$

together with its modification with environmental constraint:

$$\dot{p} = rp(1 - \frac{p}{R}).$$

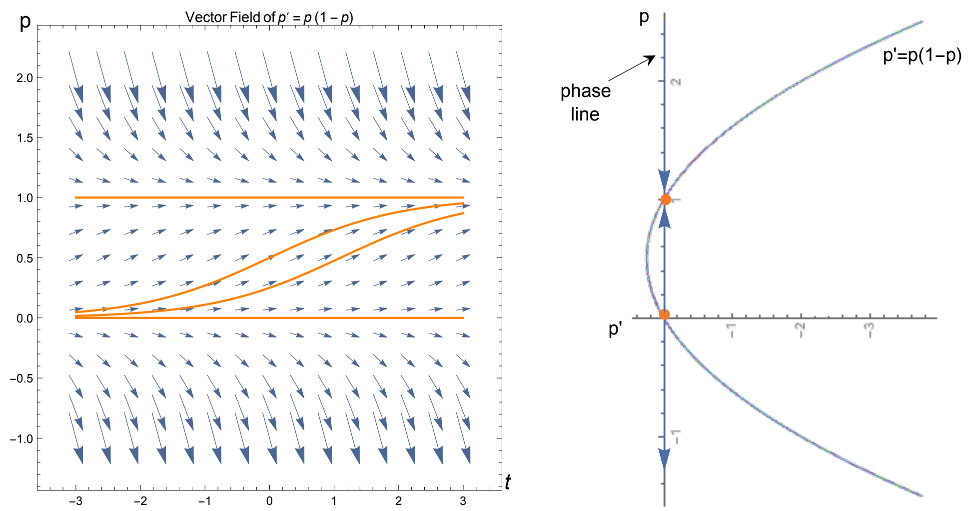
If we think of these two equations in the form

$$\dot{y} = g(t, y),$$

the function $g(t, y)$ does not depend explicitly on t. We define equations of this type as *autonomous equations*, meaning, the rate of change of the unknown function depends only on the function value and not explicitly on time. Of course, such functions always take the form

$$\dot{y} = g(y).$$

In lecture 1, we saw how plotting the vector fields of an equation may be helpful in understanding the solutions. For autonomous equations, the vector fields have the nice property that they are invariant under translation in the t-direction. So one may simply care about the slopes of the direction field along the y-axis.



Logistic Model. In the picture above, we have a plot of the vector fields of the logistic equation:

$$\dot{p} = p(1 - p),$$

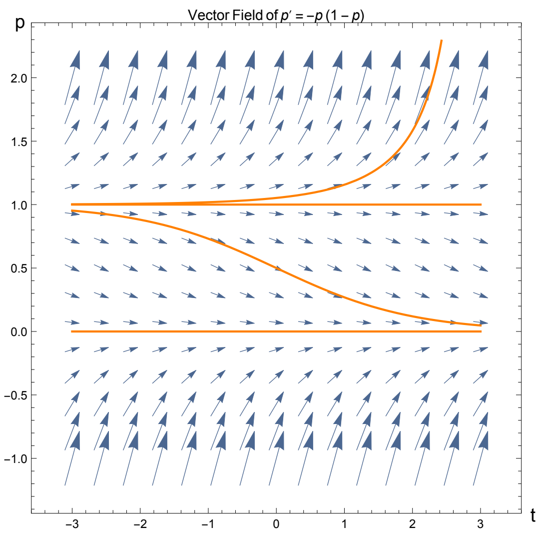
together with the graph of the function $g(p) = p(1 - p)$. Note that we have rotated the graph of $g(p) = p(1 - p)$ counter-clockwise by $90°$ so that the p axis goes parallel with that in the vector field plot. In doing this, it is easy to see that the p-intercept of the graph of $g(p) = p(1 - p)$ corresponds to the equilibrium solutions. Moreover, one could see that when p take values between $(0, 1)$, $\dot{p}$ remains positive. On the vector field side, this means that all the arrows have positive slope as long as $p \in (0, 1)$. To capture this property, we put a upward arrow between $(0, 1)$ along the p-axis. Similarly, we put a downward arrow within the interval $(1, \infty)$ along the p axis, to indicate that when p takes value in this interval, arrows in the vector field have negative slope. Again, we obtain a downward arrow in the interval $(-\infty, 0)$ along the p axis by the same argument. The p axis in the graph on the right, with all the arrows and equilibriums marked, is called the **phase line** corresponding to the logistic equation $\dot{p} = p(1 - p)$. Now, convince yourself that these arrows are capable of telling us the stability of all the equilibriums. (Do you see that along the phase line, nearby the equilibrium $p = 1$, arrows are pointing towards it? How about $p = 0$? What is your conclusion?)

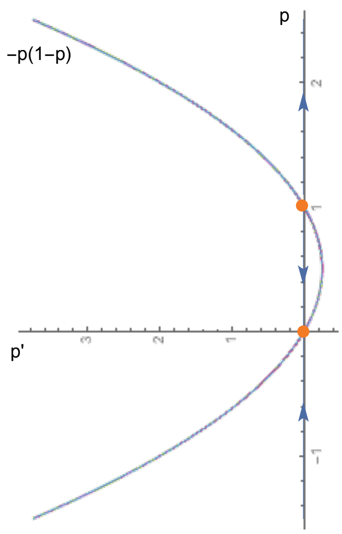
One of the reasons that we like to use phase lines to do analysis is their simplicity: (i) for an autonomous equation in the general form $\dot{y} = g(y)$, the y intercept of the graph of $g(y)$ tells us the equilibriums of the system; (ii) by figuring out the sign of $g(y)$ in each interval separated by the equilibrium points, we could draw arrows along these intervals as we did before, and thus be able to tell whether an equilibrium is stable or not.

Threshold Model. Consider the following scenario: for certain species, we do not observe exponential growth in population when the population is below certain level, but instead, the specie would more likely to go to extinction. This scenario can be modeled by the following equation:

$$\dot{p} = -rp(1 - \frac{p}{T}).$$

You may observe this equation differs from the general logistic equation only by a negative sign on the right hand side. So if we draw the graph of $g(p)$ in this case, it looks exactly like the graph of $g(p)$ in the logistic model flipped over the p-axis. Therefore, when $r = T = 1$, the phase line could be obtained simply by reversing all the arrows in the phase line of the logistic equation. In the plot below, we can see that the equilibrium $p = 0$ is stable and $p = 1$ is not.





Remark. (1) It should be pointed out here that the phase line analysis has its own limitations. For instance, in the standard threshold model with $r = T = 1$, we learned from the phase line that solutions would increase if we set an initial value to be above

1. You may guess that the solution then increases forever (for all x). However, this is not the case. By directly solving the equation using separation of variables, we could see that such solutions goes out of bound in finite time! (Try to do this as an exercise.) This is something that we could not tell from the phase line alone.

(2) Remembering the bigger picture of modeling, you may have noticed something non-realistic in the threshold model: the specie that our model describes is so "dangerous" that it will either go extinction or reach infinity amount of population, and the population $p = 1$ is unstable. A natural question is: is there a model which takes into account both the threshold and the environmental capacity? Yep, but let us see if we can construct such a model by ourselves.

Threshold & Environmental Cap: Combination. This time we draw the phase line first, based on what we would expect from the model. First, let's mark the three critical points on the p-axis: 0, T, and R ($T < R$), where T and R are the threshold and capacity. By our argument above, we wish that, for the initial conditions in the intervals $(0, T)$, (T, R), and (R, ∞), the solutions respectively descends to zero, increases to R, and decreases to R. This tells us which arrows to put in these intervals: downward for $(0, T)$ and (R, ∞), upward for (T, R). Essentially, we are looking for a function $g(p)$ that vanishes at 0, T and R. A natural choice would be

$$g(p) = \lambda p(1 - \frac{p}{T})(1 - \frac{p}{R}),$$

for some constant λ. Now in our case, we know that for $p > R$, $g(p) < 0$. Thus $\lambda < 0$. Let $-\lambda = r$. Therefore, we have obtained the model:

$$\dot{p} = -rp(1 - \frac{p}{T})(1 - \frac{p}{R}).$$

To check that this model works as expected, we plot the stream line of the vector field in the case $r = T = 1$ and $R = 3$ below.