LECTURE 23: THE OCCURRENCE OF TWO-POINT BOUNDARY VALUE PROBLEMS

As we have seen in the heat and wave equations with 1-dimensional space, or the Laplace equation with 2-dimensional space, the method of separation of variables will lead us to two ordinary differential equations. Together with the boundary conditions, we would encounter problems of the two-point boundary value type. Note that these two-point boundary value problems are quite specific in that the equation usually take the form

$$y'' + \lambda y = 0,$$

with boundary values being either

$$y(0) = y(L) = 0,$$

or

$$y'(0) = y'(L) = 0.$$

In this lecture, we will introduce a broader class of boundary value problems, generalizing both the ODEs and the boundary values we have seen previously. We will start with an example, and see how the same principle we used to solve simpler problems continues to work in less familiar, specific problems. Then we introduce the physical motivation for the broader class of equations: the Sturm-Liouville BVPs.

1. Example

Let us solve the two-point boundary value problem

$$\begin{cases} y'' + \lambda y = 0, \\ y(0) = 0, \quad y'(1) + y(1) = 0, \end{cases}$$

that is, to find all the eigenvalues and eigenfunctions of the equation.

As before, we consider three cases of λ: $\lambda = 0, \lambda < 0, \lambda > 0$. (You may ask, why not consider the case that λ is a complex number? This will be answered in the next lecture.)

- $\lambda = 0$. The general solution of the equation is

  $$y(x) = c_1 x + c_2.$$

  The boundary conditions require

  $$c_2 = 0, \quad 2c_1 + c_2 = 0,$$

  which is

  $$c_1 = c_2 = 0.$$

  Hence, $\lambda = 0$ is not an eigenvalue of the problem.

Date: 06/17/15.

- $\lambda < 0$. Let $\lambda = -\mu^2$, where $\mu > 0$. The general solution of the equation is
$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}.$$
The boundary values require,
$$\begin{cases} c_1 + c_2 = 0, \\ \mu(c_1 e^{\mu} - c_2 e^{-\mu}) + c_1 e^{\mu} + c_2 e^{-\mu} = 0. \end{cases}$$
In matrix form, this is
$$\begin{bmatrix} 1 & 1 \\ (\mu+1)e^{\mu} & (1-\mu)e^{-\mu} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0.$$
There exists a nonzero pair (c_1, c_2) so that the above equality is satisfied if and only if the determinant of the coefficient matrix vanishes, which is equivalent to
$$\mu = \frac{e^{-\mu} - e^{\mu}}{e^{\mu} + e^{-\mu}}.$$
However, this never holds for $\mu > 0$, since the right hand side is negative, by the monotonicity of the exponential function.
- $\lambda > 0$. Let $\lambda = \mu^2$, $\mu > 0$. The general solution of the equation is
$$y(x) = c_1 \cos \mu x + c_2 \sin \mu x.$$
The boundary values require
$$\begin{cases} c_1 = 0, \\ -c_1 \mu \cos \mu + c_2 \mu \cos \mu + c_1 \cos \mu + c_2 \sin \mu = 0. \end{cases}$$
It is easy to see that λ is an eigenvalue if and only if
$$\mu \cos \mu + \sin \mu = 0.$$
Note that this equality cannot hold unless $\cos \mu \neq 0$, thus it is equivalent to
$$\mu = -\tan \mu.$$
This tells us that the values of μ whose square are eigenvalues are exactly the x-coordinates of the intersection points of the graphs of
$$y = x,$$
and
$$y = -\tan x.$$

Exercise. Plot the graphs of $y = x$ and $y = -\tan x$ and see that their intersection points are infinitely many. If we number the x-coordinate of these points from left to right as $\mu_n (n = 1, 2, 3, ...)$ then it can be observed that
$$\mu_n \approx \frac{2n-1}{2}\pi,$$
when n is large enough. (Note: For large n, μ_n is large, therefore $-\tan \mu_n$ is large, which means μ_n must be close to an odd multiple of $\frac{\pi}{2}$ from the right.)

We conclude that the eigenvalues of the original two-point BVP are

$$\lambda_n = \mu_n^2,$$

where $\mu_n = -\tan \mu_n$, $(0 < \mu_1 < \mu_2 < ...)$, and the eigenfunctions are

$$y_n(x) = \sin \mu_n x.$$

2. Arise of Sturm-Liouville BVPs

In the derivation of the heat equation in a rod (Lecture 19), we introduced the constants:

- λ: measuring the capability of the material to conduct heat;
- μ: measuring the capability of the material to restore heat;
- A: area of the cross section of the rod.

We assumed that there is *no heat loss from the side (cylindrical) boundary* of the rod. Also, we considered only two kinds of boundary values: zero/constant temperature at both ends, insulated ends.

In a more general setting, the rod may not be made uniformly of the same material, and the cross section may not be a constant. Thus, λ, μ, A now may depend on x. Also, closer to reality, we could assume that the heat loss rate $\ell(x)$ (per unit length) through the side boundary to be proportional (for each x value) to the temperature $u(x,t)$. In this case, a similar analysis as before leads to

$$\Delta H = -A(x)\lambda(x)\frac{\partial u}{\partial x}(x,t)\Delta t + A(x+\Delta x)\lambda(x+\Delta x)\frac{\partial u}{\partial x}(x+\Delta x,t)\Delta t - \ell(x)u(x,t)\Delta x\Delta t$$
$$= \mu(x)\frac{\partial u}{\partial t}(x,t)\Delta t\Delta x.$$

Passing to the limit $\Delta x \to 0$, we could obtain

$$[A(x)\lambda(x)u_x(x,t)]_x - \ell(x)u(x,t) = \mu(x)u_t(x,t).$$

On the other hand, you can check that the boundary condition

$$\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = 0, \qquad \beta_1 u(L,t) + \beta_2 u_x(L,t) = 0$$

generalizes the two kinds of boundary values we have studied before: $u(0,t) = u(L,t) = 0$ and $u_x(0,t) = u_x(L,t) = 0$, by setting values to $\alpha_1, \alpha_2, \beta_1, \beta_2$.

To put in a more concise form, we are now considering the boundary-value problem:

$$\text{(I)} \begin{cases} [p(x)u_x]_x - q(x)u = r(x)u_t, & t > 0, 0 < x < L, \\ \alpha_1 u(0,t) + \alpha_2 u_x(0,t) = 0, \qquad \beta_1 u(L,t) + \beta_2 u_x(L,t) = 0, & t > 0. \end{cases}$$

To solve (I), we first look for solutions of the form

$$u(x,t) = X(x)T(t).$$

Plugging in the equation gives us

$$r(x)XT' = [p(x)X']'T - q(x)XT,$$

which is equivalent to

$$\frac{T'}{T} = \frac{[p(x)X']'}{r(x)X} - \frac{q(x)}{r(x)}.$$

By a familiar argument, we know that both the LHS and RHS in the equality above are equal to a same constant $-\lambda$. Thus,

$$\begin{cases} [p(x)X']' - q(x)X + \lambda r(x)X = 0, & 0 < x < L, \\ T' + \lambda T = 0, & T > 0. \end{cases}$$

Plugging $u(x,t) = X(x)T(t)$ in the boundary conditions, it is easy to see that $X(x)$ satisfies

$$\alpha_1 X(0) + \alpha_2 X'(0) = 0, \qquad \beta_1 X(L) + \beta_2 X'(L) = 0.$$

Now, note that given λ, $T(t)$ is evident from the equation it satisfies. So our central problem is finding eigenvalues and eigenfunctions of the two-point boundary value problem that $X(x)$ satisfies:

$$\begin{cases} [p(x)X']' - q(x)X + \lambda r(x)X = 0, & 0 < x < L, \\ \alpha_1 X(0) + \alpha_2 X'(0) = 0, & \beta_1 X(L) + \beta_2 X'(L) = 0. \end{cases}$$

For simplicity, one could scale the length L to be 1 by changing the units. Also, in order for the boundary conditions to be valid, we assume $(\alpha_1, \alpha_2) \neq (0,0)$ and $(\beta_1, \beta_2) \neq (0,0)$. Thus, the two-point BVP for $X(x)$ is of the form

$$\text{(SL)} \begin{cases} [p(x)y']' - q(x)y + \lambda r(x)y = 0, & 0 < x < 1, \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0, & \beta_1 y(1) + \beta_2 y'(1) = 0, \end{cases}$$

which we call a *Sturm-Liouville* boundary value problem.