LECTURE 22: THE LAPLACE EQUATION

Previously, we have studied the heat equation with 1-dimensional space (i.e., the equation of heat conduction in a rod):

$$\alpha^2 u_{xx} = u_t.$$

It can be derived that when the dimension of the space is two or three (think of a round disk or a solid ball in each case), the heat equations are

$$\alpha^2(u_{xx} + u_{yy}) = u_t,$$

and

$$\alpha^2(u_{xx} + u_{yy} + u_{zz}) = u_t,$$

respectively.

We can ask, what is the steady state of the heat distribution? That is, given certain *constant* boundary condition, what is the solution of the heat equation which does not depend on t, i.e., $u_t = 0$? In particular, we have in mind the **Dirichlet Problem**, and we focus on the case of two dimensional space:

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in \Omega, \\ u(x, y) = f(x, y), & (x, y) \in \partial\Omega, \end{cases}$$

where Ω denotes the two dimensional domain and $\partial\Omega$ is its boundary. (For instance, when Ω is a round disk, then $\partial\Omega$ is its boundary circle, etc.)

In this lecture, we focus on two kinds of domains:

- Ω is a rectangle, its boundary consisting of four line segments.
- Ω is a round disk of radius a, its boundary being the circle of radius a and the same center.

1. Domain $\Omega = [0, a] \times [0, b]$

In this case, because the boundary of the domain consists of four line segments, the Dirichlet problem in general takes the form

$$\begin{cases} u_{xx} + u_{yy} = 0, & & (x, y) \in (0, a) \times (0, b), \\ u(x, 0) = g_1(x), & u(x, b) = g_2(x), & x \in (0, a), \\ u(0, y) = f_1(y), & u(a, y) = f_2(y), & y \in (0, b). \end{cases}$$

However, we claim that to solve this boundary value problem, we need only to solve for the four simpler boundary conditions:

$$\begin{array}{llll} (1)\ u(x,0) = g_1(x), & u(x,b) = 0, & u(0,y) = 0, & u(a,y) = 0; \\ (2)\ u(x,0) = 0, & u(x,b) = g_2(x), & u(0,y) = 0, & u(a,y) = 0; \\ (3)\ u(x,0) = 0, & u(x,b) = 0, & u(0,y) = f_1(y), & u(a,y) = 0; \\ (4)\ u(x,0) = 0, & u(x,b) = 0, & u(0,y) = 0, & u(a,y) = f_2(y). \end{array}$$

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Exercise. Convince yourself that this claim is valid. Hint: Note that the equation is linear and consider superposing the solutions of the four sub-BVP's.

Therefore, we only focus on one of the four listed boundary values, say the fourth one, and consider solving

$$\begin{cases} u_{xx} + u_{yy} = 0, & (x, y) \in (0, a) \times (0, b), \\ u(x, 0) = 0, \quad u(x, b) = 0, & x \in (0, a), \\ u(0, y) = 0, \quad u(a, y) = f(y), & y \in (0, b). \end{cases}$$

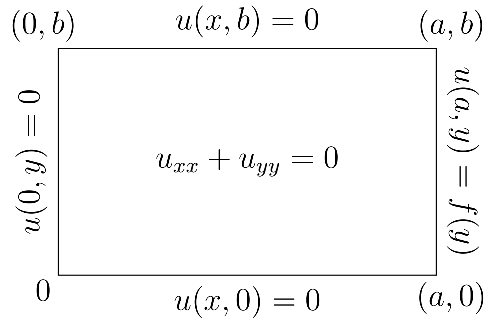

FIGURE 1. Laplace Equation with Dirichlet Boundary Condition (4).

1.1. **Separation of Variables.** Consider solution of the form

$$u(x, y) = X(x)Y(y).$$

Plugging in the Laplace equation, and by a familiar argument, we obtain

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda,$$

where λ is some constant. This gives us two ODEs,

$$\begin{cases} X'' - \lambda X &= 0, \\ Y'' + \lambda Y &= 0. \end{cases}$$

1.2. **Homogeneous Boundary Conditions.** Now consider the three pieces of boundary conditions which are homogeneous. We have

$$X(x)Y(0) = X(0)Y(y) = X(x)Y(b) = 0,$$

for all possible x, y. Therefore,

$$Y(0) = Y(b) = 0, \qquad X(0) = 0.$$

Combined with the two ODEs we obtained using separation of variables, we can see that $Y(y)$ solves the two-point boundary value problem

$$\begin{cases} Y'' + \lambda Y = 0, \\ Y(0) = Y(b) = 0. \end{cases}$$

Once more, we are in a familiar situation. And we could claim that the eigenvalues are

$$\lambda = \left(\frac{n\pi}{b}\right)^2, \qquad n = 1, 2, 3, ...$$

and the eigenfunctions are

$$Y_n(y) = \sin \frac{n\pi}{b} y.$$

Further more, for each n and the corresponding $\lambda = \frac{n^2\pi^2}{b^2}$ we have for $X(x)$:

$$\begin{cases} X'' - \frac{n^2\pi^2}{b^2} X = 0, \\ X(0) = 0. \end{cases}$$

The equation (in X) itself has the general solution

$$X_n(x) = k_1 e^{\frac{n\pi}{b}x} + k_2 e^{-\frac{n\pi}{b}x},$$

for some constants k_1, k_2. Then the boundary value $X_n(0) = 0$ requires

$$k_1 + k_2 = 0.$$

Therefore,

$$X_n(x) = k_1 e^{\frac{n\pi}{b}x} - k_1 e^{-\frac{n\pi}{b}x},$$

and, up to a multiplicative constant, this is just

$$X_n(x) = \sinh\left(\frac{n\pi x}{b}\right).$$

Now we have obtained

$$u_n(x,y) = X_n(x)Y_n(y) = \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right), \qquad n = 1, 2, 3, ...$$

1.3. **Non-homogeneous Boundary Value.** Finally, we consider the superposition of $u_n(x,y)$, which, as usual, solves the linear equation together with the three pieces of *homogeneous* boundary conditions.

$$u(x,y) = \sum_{n=1}^{\infty} c_n u_n(x,y).$$

The last non-homogeneous piece of the boundary values requires

$$u(a,y) = f(y) = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi a}{b} \sin\frac{n\pi y}{b}.$$

Therefore, noting that $f(y)$ is expanded as a sine series, we have the coefficients c_n:

$$c_n = \left(\sinh\frac{n\pi a}{b}\right)^{-1} \cdot \frac{2}{b} \int_0^b f(y) \sin\frac{n\pi y}{b} dy.$$

This completes the formal solution of the Dirichlet problem with boundary condition (4).

2. Domain $\Omega = B(0,a)$

Let $B(0,a)$ denote the disk centered at the origin with radius $a > 0$. Using polar coordinates, the Laplace equation $u_{xx} + u_{yy} = 0$ can be rewritten as

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0.$$

Exercise. Check the previous statement about the equivalent form of the Laplace equation in different coordinate systems.

In this case, the Dirichlet boundary value problem takes the form

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & r \in [0,a), \theta \in \mathbb{R}, \\ u(a,\theta) = f(\theta), & \theta \in \mathbb{R}, \end{cases}$$

for some function $f(\theta)$.

Particularly, we should note that $u(x,\theta)$ represents the distribution of temperature in a disk, it must be periodic in θ with period 2π. Same for $f(x)$. In symbols, this is just

$$f(\theta) = f(\theta + 2\pi), \qquad \theta \in \mathbb{R}.$$

$$u(x,\theta) = u(x, \theta + 2\pi), \qquad x \in [0,a), \theta \in \mathbb{R}.$$

Moreover, the temperature cannot be infinite, thus we also require

$$u(x,\theta) < \infty$$

for any valid solution of the Dirichlet problem

Now we proceed to solve the equation.

2.1. **Separation of Variables.** Consider solutions of the form

$$u(r,\theta) = R(r)\Theta(\theta).$$

Plugging in the equation, we have

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0,$$

By a familiar argument, this leads to

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda,$$

for some constant λ.

Thus, we have two ODEs to consider:

$$\begin{cases} r^2R'' + rR' - \lambda R = 0, \\ \Theta'' + \lambda\Theta = 0. \end{cases}$$

In particular, note that the ODE in $R(r)$ is an Euler equation.

2.2. **Valid Solutions.** As noted before, a solution is valid only if $\Theta(\theta)$ is periodic with period 2π. On the other hand, the equation that Θ satisfies gives the three possibilities:

- $\lambda = -\mu^2 < 0$. The only solutions are in the form

  $$\Theta(\theta) = c_1e^{\mu\theta} + c_2e^{-\mu\theta},$$

  which is not periodic unless $c_1 = c_2 = 0$. Thus, there are no negative eigenvalues.
- $\lambda = 0$. Again, periodicity requires $\Theta(\theta)$ to be a constant. On the other hand, we have, by the solutions of the Euler equation

  $$r^2R'' + rR' = 0,$$

  that

  $$R(r) = k_1 + k_2 \ln r.$$

This time, by the boundedness of solutions, we must have $k_2 = 0$, since $\ln r \to -\infty$ as $r \to 0^+$. Hence, $R(r)$ is also a constant. We conclude that $\lambda = 0$ is an eigenvalue and the only eigenfunction (up to a multiplicative constant) is

$$u_0(r,\theta) = \frac{1}{2}.$$

- $\lambda = \mu^2 > 0$. The ODEs now become

$$\begin{cases} r^2R'' + rR' - \mu^2 R = 0, \\ \Theta'' + \mu^2\Theta = 0. \end{cases}$$

We have the general solutions

$$\begin{aligned} R(r) &= k_1 r^{\mu} + k_2 r^{-\mu}, \\ \Theta(\theta) &= d_1 \sin\mu\theta + d_2\cos\mu\theta, \end{aligned}$$

for arbitrary constants k_1, k_2, d_1, d_2.

Again, the boundedness of solutions forces $k_2 = 0$, and that the $\Theta(\theta)$ must have 2π as a period requires μ to be a positive integer. Therefore, we can choose

$$\begin{aligned} R_n(t) &= r^n, \\ \Theta_n(\theta) &= d_1\sin n\theta + d_2\cos n\theta. \end{aligned}$$

And

$$u_n(r,\theta) = r^n(p_n\sin n\theta + q_n\cos n\theta), \qquad n = 1,2,3,...$$

2.3. **Boundary Condition.** Now, we consider a superposition of the eigenfunctions and find the coefficients so that the superposition fits the boundary value. First, let

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n(p_n\sin n\theta + q_n\cos n\theta).$$

Then the boundary value requires

$$u(a,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a^n(p_n\sin n\theta + q_n\cos n\theta).$$

Since $f(\theta)$ has the period 2π, we have the coefficients

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta, \\ p_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta, \qquad n = 1,2,3,... \\ q_n &= \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta. \qquad n = 1,2,3... \end{aligned}$$

This, formally, completes solving the Dirichlet problem on a round disk. Of course, as in all the BVP's we have studied, one could ask about the convergence of the solution and whether the formal solution converges at the boundary to the exact boundary condition. In fact, the answer is not hard to formulate, according to the Fourier convergence theorem.