LECTURE 21: WAVE EQUATIONS, VIBRATION OF ELASTIC STRING

1. Standing and Traveling Waves

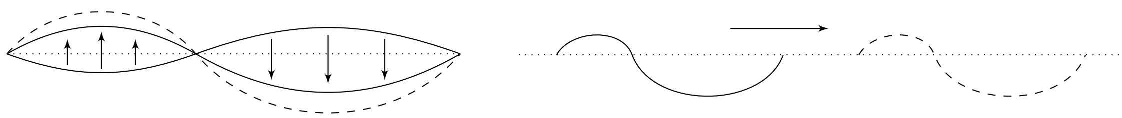
Speaking about waves, there are two basic types: *standing waves* and *traveling waves*. The former, as time evolves, the configuration of the wave changes by an amplifying factor. To be precise, we have

$$u(x,t) = X(x)T(t),$$

in which $u(x,t)$ denote the displacement of the particle at coordinate x of the string at time t from its resting position. The latter, as time evolves, does not change the configuration of the wave, but only travels in a certain direction at a certain velocity. In symbols, this is just

$$u(x,t) = F(x-ct),$$

which means that wave, with configuration being the graph of $F(x)$, is traveling to the right at a velocity of c. Graphically, these two types of waves are illustrated below:



(A) Standing Wave (B) Traveling Wave

Figure 1. Two Types of Waves

For the wave equation that we are about to describe, it turns out that the standing and traveling waves will give us two different views of its solution.

2. The Wave Equation $c^2 u_{xx} = u_{tt}$

Given an elastic string of length L, we could imagine that the mass is concentrated on certain particles along the string (x_n, with $x_{n+1} - x_n = \Delta x$), and that each particle is allowed to move vertically (Figure 2).

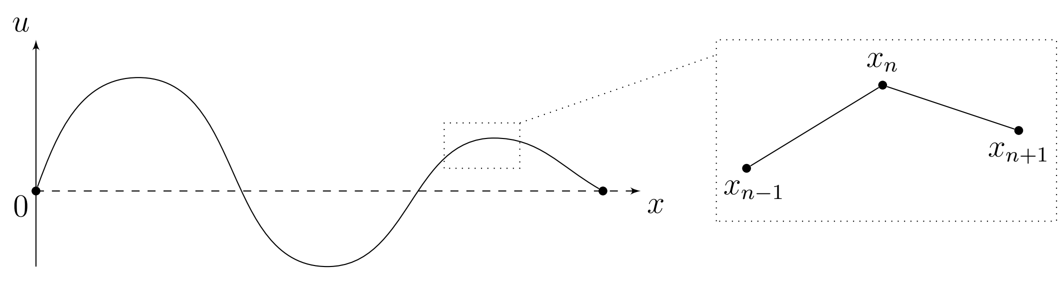


Figure 2. Elastic String

Date: 06/15/15.

Furthermore, we assume that the tension along the string is a constant τ. Therefore, the vertical component of the tension of the string connecting to x_n, left and right, are respectively

$$F_\ell = -\tau \frac{u(x_n,t) - u(x_{n-1},t)}{\Delta x}, \qquad F_r = \tau \frac{u(x_{n+1},t) - u(x_n,t)}{\Delta x}.$$

Hence, ignoring gravity, the force of the string at the particle x_n is

$$F = F_\ell + F_r = \tau \frac{u(x_n + \Delta x, t) - 2u(x_n,t) + u(x_n - \Delta x, t)}{\Delta x}.$$

By Newton's second law, we have

$$F = ma = \Delta x \rho u_{tt}(x_n,t),$$

where ρ is the density of the string. This gives us

$$u_{tt}(x_n,t) = \frac{\tau}{\rho} \frac{u(x_n + \Delta x, t) - 2u(x_n,t) + u(x_n - \Delta x, t)}{\Delta x^2}.$$

Taking the limit $\Delta x \to 0$ and using the Taylor expansion at $x = x_n$ (or using the L'Hôpital's rule twice), we obtain the equality

$$u_{tt}(x_n,t) = \frac{\tau}{\rho} u_{xx}(x_n,t).$$

Indeed, there is nothing special about the position of x_n here, so let it be denoted by x instead ($0 < x < L$). Also let us call τ/ρ as c^2. The wave equation follows:

$$u_{tt}(x,t) = c^2 u_{xx}(x,t), \qquad 0 < x < L, t > 0.$$

One may ask: what is c? In section 4, we will see that the solution of this equation is the superposition of two traveling waves, one to the left, the other to the right, both with velocity c.

3. Solution of Initial-Boundary Value Problems

Pick an elastic string of length L, hold it tight at the both ends, then pull the middle of the string away from its resting position, release it from static, we'll observe an unique motion of the string over time. Mathematically, the condition we have put on the string are exactly the following:

- Fixed ends: $u(0,t) = u(L,t) = 0$.
- Pulling the string from the resting position: $u(x,0) = f(x)$.
- String released from static: $u_t(x,0) = 0$.

Clearly, this is boundary values together with two pieces of initial values (since the equation has the second derivative with respect to t).

More generally, we could consider the following initial-boundary value problem

$$(\mathrm{I}) \begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), \ u_t(x,0) = g(x), & 0 \le x \le L. \end{cases}$$

Furthermore, this problem can be *reduced* to two simpler problems, one with zero initial velocity, the other with zero initial value.

$$\text{(II)}\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x),\ u_t(x,0) = 0, & 0 \le x \le L. \end{cases}$$

$$\text{(III)}\begin{cases} u_{tt} = c^2 u_{xx}, & 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = 0,\ u_t(x,0) = g(x), & 0 \le x \le L. \end{cases}$$

Exercise. Show that if $u_1(x,t)$ is a solution of (II) and $u_2(x,t)$ is a solution of (III), then $u_1(x,t) + u_2(x,t)$ is a solution of (I), assuming that $f(x), g(x)$ in (I) are the same as those in (II) and (III).

3.1. **Zero Initial Velocity.** That is, equation (II). Its solution is similar to that of the heat equations. First consider separation of variables, then use boundary values to narrow down the forms of the solution, finally, consider a superposition of solutions which also satisfies the initial values.

3.1.1. *Separation of Variables.* Let

$$u(x,t) = X(x)T(t).$$

If this solve the wave equation, we must have

$$X(x)T''(t) = c^2 X''(x)T(t),$$

that is,

$$\frac{X''}{X}(x) = \frac{T''}{c^2 T}(t).$$

As before, we know that the value of this expression is a constant, since it does not depend on t, nor x. So, let

$$\frac{X''}{X}(x) = \frac{T''}{c^2 T}(t) = -\lambda,$$

for some constant λ . Written in two ODEs, this is just

$$\begin{cases} X'' + \lambda X = 0, \\ T'' + c^2 \lambda T = 0. \end{cases}$$

3.1.2. *Boundary Values.* First, the fixed ends gives us

$$u(0,t) = u(L,t) = 0,$$

which, under our assumption of the form of solution $u(x,t) = X(x)T(t)$, is just

$$X(0)T(t) = X(L)T(t) = 0,$$

leading to

$$X(0) = X(L) = 0.$$

With the equation

$$X'' + \lambda X = 0,$$

it is an easy exercise of two-point boundary problem to show that the only eigenvalues are

$$\lambda = \left(\frac{n\pi}{L}\right)^2,$$

and correspondingly,

$$X_n(x) = \sin \frac{n\pi}{L} x, \qquad T_n(t) = a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t, \quad n = 1, 2, 3, ...$$

where a_n, b_n are arbitrary constants.

3.1.3. *Initial Values.* Now we consider the initial values. In particular, if we let $u_n(x,t) = X_n(x)T_n(t)$, then

$$(u_n)_t(x, 0) = 0$$

implies

$$T_n'(0) = 0,$$

hence

$$b_n = 0$$

and, up to multiplication by a constant, we can choose $T_n(t) = \cos \frac{n\pi c}{L} t$ and thus

$$u_n(x,t) = \sin \frac{n\pi}{L} x \; \cos \frac{n\pi c}{L} t.$$

Finally, we note that any *superposition* of the u_n's would satisfy the initial-boundary value problem (II), except for one piece of the initial value: $u(x, 0) = f(x)$. Therefore, we let

$$u(x,t) = \sum_{n=1}^{\infty} d_n u_n(x,t) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi}{L} x \; \cos \frac{n\pi c}{L} t.$$

Hence, by the initial value, we have

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} d_n \sin \frac{n\pi}{L} x.$$

The Fourier coefficients a sine series gives

$$d_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx.$$

This completes the solving (II).

3.2. **Zero Initial Position.** That is, consider the initial-boundary value problem (III). Before reading the solution below, you may consider trying to solve it as an exercise, as it is quite similar to the preceding case. In fact, nothing is different in the steps of separation of variables and considering the boundary values. There we had

$$X_n(x) = \sin \frac{n\pi}{L} x, \qquad T_n(t) = a_n \cos \frac{n\pi c}{L} t + b_n \sin \frac{n\pi c}{L} t, \quad n = 1, 2, 3, ...$$

where a_n, b_n are arbitrary constants.

Now the initial value is different. The homogeneous condition is at the position, which requires

$$u(x, 0) = 0.$$

Similarly as before, convince yourself that this leads to $a_n = 0$, and, un to multiplication by a constant,

$$T_n(t) = \sin \frac{n\pi c}{L} t.$$

Therefore,we let

$$u_n(x,t) = X_n(x)T_n(t) = \sin \frac{n\pi}{L} x \; \sin \frac{n\pi c}{L} t.$$

And for each $n = 1, 2, 3, ...$, $u_n(x,t)$ satisfies (III) except for the condition $u_t(x,0) = g(x)$.

Finally, we superpose the u_n's, and set

$$u(x,t) = \sum_{n=1}^{\infty} k_n u_n(x,t) = \sum_{n=1}^{\infty} k_n \sin\frac{n\pi}{L}x \ \sin\frac{n\pi c}{L}t.$$

By $u_t(x,0) = g(x)$,

$$\sum_{n=1}^{\infty} \frac{n\pi c}{L} k_n \sin\frac{n\pi}{L}x = g(x).$$

Therefore, the coefficients k_n are determined as

$$k_n = \frac{L}{n\pi c}\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x dx.$$

This completes solving (III).

4. Traveling Waves and D'Alembert's Formula

In this section, we look at the solutions of the wave equation

$$c^2 u_{xx} = u_{tt}$$

from a different angle. First, let us consider the new variables

$$\xi = x - ct, \qquad \eta = x + ct.$$

Exercise. Show that under the new variables, the wave equation above is equivalent to

$$u_{\xi\eta} = 0.$$

By the fundamental theorem of calculus (use it twice!), we know that any solution of this equation must be of the form

$$u(\xi,\eta) = F(\xi) + G(\eta),$$

for some functions F, G. (And, in fact, for any differentiable functions $F(x)$ and $G(x)$, $F(\xi) + G(\eta)$ solves the wave equation.)

Therefore, change the variables back to x, t in the solution, we have that any solution of the wave equation must be of the form

$$u(x,t) = F(x - ct) + G(x + ct),$$

which is simply the superposition of two *traveling waves*, one going to the left, the other to the right, both of velocity c.

Furthermore, if we also know initial conditions, say,

$$u(x,0) = f(x), \quad u_t(x,0) = g(x),$$

then the solution is given by the **d'Alembert's Formula**:

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau.$$