LECTURE 20: MORE HEAT CONDUCTION PROBLEMS

Following from the last lecture, we consider two more settings for the problem of heat conduction in a rod. In one setting, instead of assuming the temperature at both ends of the rod to be zero, we assume that they are some constants. In another, we assume that both ends of the rod are insulated, that is, there is no heat conduction there. Using symbols, the first setting can be simply described as

$$u(0,t) = T_1, \quad u(L,t) = T_2,$$

for some constants T_1, T_2. The second setting is just

$$\frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(L,t) = 0.$$

1. Constant Boundary Values

The initial-boundary value problem is now

$$\text{(I)}\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, t > 0, \\ u(0,t) = T_1, u(L,t) = T_2, & t > 0, \\ u(x,0) = f(x), & 0 \leq x \leq L. \end{cases}$$

To solve it, first suppose that some function $v(x,t)$ satisfies

$$\text{(II)}\begin{cases} \alpha^2 v_{xx} = v_t, & 0 < x < L, t > 0, \\ v(0,t) = T_1, v(L,t) = T_2, & t > 0. \end{cases}$$

Then, it is clear by the linearity of the equation that $w(x,t) = u(x,t) - v(x,t)$ would satisfy

$$\text{(III)}\begin{cases} \alpha^2 w_{xx} = w_t, & 0 < x < L, t > 0, \\ w(0,t) = 0, w(L,t) = 0, & t > 0, \\ w(x,0) = f(x) - v(x,0), & 0 \leq x \leq L, \end{cases}$$

which is an initial-boundary value problem with zero boundary values (We know how to solve this!). In other words, to solve (I), we need only to find a particular solution $v(x,t)$ to (II), then solve (III), obtaining $w(x,t)$. Finally, $u(x,t) = v(x,t) + w(x,t)$ is a solution to (I).

From the analysis above, we can see that the effectiveness of the idea relies on how easy it is to find a particular solution of (II). In fact, this task is quite simple: we do not even need $v(x,t)$ to depend on t. Suppose that $v(x,t) = v(x)$ is a solution of (II). We have

$$\alpha^2 v'' = 0, \qquad v(0) = T_1, v(L) = T_2.$$

Thus immediately,

$$v(x) = ax + b, \qquad v(0) = T_1, v(L) = T_2.$$

We have

$$a = \frac{T_2 - T_1}{L}, \qquad b = T_1,$$

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and
$$v(x) = \frac{T_2 - T_1}{L} x + T_1.$$

In essence, we have completed our theory for solving the initial-boundary value problem (I). However, it is always instructive to ask: why do we think of finding a particular solution of (II) at all? The answer is perhaps, again, *superposition*. This really isn't something new to us. Maybe you could remember from the theory of ODEs. To solve a non-homogeneous linear equation, we find a particular solution first; then *superpose* with any homogeneous solution to get general solutions. There, we did not use the exact word "superpose", but the idea is completely similar. Finally, we stress that it is the *linearity of the equation* that enables such superposition.

Remark. The function $v(x)$ is sometimes called a *static solution* of (II), which literally means: does not vary with time.

2. Rod with Insulated Ends

In this case, the initial-boundary value problem is
$$\text{(IV)} \begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, t > 0, \\ u_x(0,t) = 0, u_x(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 \le x \le L. \end{cases}$$
Solution of this problem is completely analogous to the one with zero boundary values. So you might want to solve this as an exercise to test your understanding of the previous lecture. The crucial steps are still: (1) separation of variables; (2) taking into account the boundary values; (3) using the principle of superposition to fit the initial condition. Let's do it.

2.1. **Separation of Variables.** Let
$$u(x,t) = X(x)T(t)$$
for some functions $X(x), T(t)$. Thus, plugging in the equation, we could obtain
$$\frac{X''}{X}(x) = \frac{T'}{\alpha^2 T}(t),$$
which does not depend on t not x, hence a constant, say, $-\lambda$. Now we have
$$\begin{cases} X'' + \lambda X = 0, \\ T' + \alpha^2 \lambda T = 0. \end{cases}$$

2.2. **Boundary Values.** For $u(x,t) = X(x)T(t)$, the boundary values are
$$X'(0)T(t) = X'(L)T(t) = 0.$$
For non-zero $T(t)$, this is just
$$X'(0) = X'(L) = 0.$$
Therefore, $X(x)$ is satisfying
$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0. \end{cases}$$

As before, this is treated as a two-point boundary value problem. The general solution of the equation $X'' + \lambda X = 0$ is

$$X(x) = \begin{cases} c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x, & \lambda > 0, \\ c_1 x + c_2, & \lambda = 0, \\ c_1 e^{\sqrt{-\lambda} x} + c_2 e^{\sqrt{-\lambda} x}, & \lambda < 0. \end{cases}$$

Considering the boundary values $X'(0) = X'(L) = 0$, it is easy to check that c_1, c_2 are not both zero only when

- Case 1:

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = 1, 2, 3, ...$$

and the corresponding nonzero solutions are, up to multiplication by a constant,

$$X_n(x) = \cos \frac{n\pi x}{L}.$$

On the other hand, for each value of n, we can choose

$$T_n(t) = e^{-\lambda \alpha^2 t} = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}.$$

- Case 2:

$$\lambda = 0,$$

and the solutions $X(x)$ are, up to multiplication by a constant,

$$X_0(x) = \frac{1}{2}.$$

In this case, we can choose

$$T_0(t) = 1.$$

Therefore, the boundary value problem

$$\text{(V)} \begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, t > 0, \\ u_x(0,t) = 0, u_x(L,t) = 0, & t > 0 \end{cases}$$

has solutions

$$u_0(x,t) = X_0(x) T_0(t) = \frac{1}{2},$$

and

$$u_n(x,t) = X_n(x) T_n(t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \cos \frac{n\pi x}{L}, \qquad n = 1, 2, 3, ...$$

2.3. **Initial Value.** Again, by the linearity of the equation, the principle of superposition tells us that

$$u(x,t) = \sum_{m=0}^{\infty} a_m u_m(x,t)$$

is a solution of the boundary value problem (V) for any a_n, as long as the limit of the sum exists.

Furthermore, considering the initial condition, we need

$$u(x,0) = \sum_{m=0}^{\infty} a_m u_m(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} = f(x).$$

In other words, a_n are the coefficients in the expansion of $f(x)$ as a cosine series. Therefore,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \qquad m = 0, 1, 2, 3, ...$$

and, formally,

$$u(x,t) = \sum_{m=0}^{\infty} a_m u_m(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}\cos\frac{n\pi x}{L}$$

is the solution of the initial-boundary value problem (IV).