LECTURE 2: SEPARABLE EQUATIONS, EXISTENCE AND UNIQUENESS THEOREMS

1. Separable Equations

Last time we learned how to solve first order linear ordinary differential equations. Now we attempt to climb up the ladder of the classification of equations a bit, by removing the requirement that the equations are linear.

Recall that a first order ODE, by definition, can always be written in the form:

$$f(x, y, \frac{dy}{dx}) = 0,$$

where the function $f(x, y, \frac{dy}{dx})$ may not be linear in $y, \frac{dy}{dx}$. Of course, one could easily come up with a list:

$$\begin{aligned}
\left(\frac{dy}{dx}\right)^2 - 5y + 4x &= 0, \\
\sin\frac{dy}{dx} + x - 2 &= 0, \\
\left(\frac{dy}{dx}\right)^5 + y\frac{dy}{dx} + \ln\frac{dy}{dx} + y &= 0, \\
\frac{dy}{dx} - e^{xy} &= 0, \\
\frac{dy}{dx} - \frac{x^2+1}{\sin y + 2} &= 0.
\end{aligned}$$

In this list, observe that the first two equations are non-linear in $\frac{dy}{dx}$, but we are lucky to be able to solve the equations algebraically for $\frac{dy}{dx}$. For example, the first equation above is equivalent to

$$\frac{dy}{dx} = \pm(5y - 4x)^{\frac{1}{2}},$$

and the second equation is equivalent to

$$\frac{dy}{dx} = \sin^{-1}(2 - x).$$

Unfortunately, you will find it hard to apply the same argument to the third equation above. But, ideally, let us restrict to equations of the form

$$\frac{dy}{dx} = g(x, y),$$

for now.

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Returning to the list and focusing on the last equation, observe that this equation is just

$$(\sin y + 2)\frac{dy}{dx} = x^2 - 1.$$

The left hand side of this equality reminds us of the chain rule,

$$\frac{d}{dx}f(y(x)) = \frac{d}{dy}f(y)\frac{dy}{dx}.$$

In our case, it is evident that $f(y) = -\cos y + 2y + C$ for some constant C. On the other hand, note that the right hand side is

$$x^2 - 1 = \frac{d}{dx}(\frac{1}{3}x^3 - x + D),$$

for some constant D. So the original equation can be written as

$$\frac{d}{dx}(-\cos y + 2y + C) = \frac{d}{dx}(\frac{1}{3}x^3 - x + D),$$

that is,

$$\frac{d}{dx}(\frac{1}{3}x^3 - x + \cos y - 2y + D - C) = 0.$$

It follows that

$$\frac{1}{3}x^3 - x + \cos y - 2y + D - C = E,$$

for arbitrary constants C, D, E. Grouping the constants together and call it C, we have

$$\frac{1}{3}x^3 - x + \cos y - 2y + C = 0.$$

Two remarks.

First, the expression we obtained here is not in the form $y = y(x)$, but an equation (non-differential) relating x, y. Fixing C, this relation is represented by a piece of curve in the x-y plane. Be careful that sometimes a curve in the x-y plane can not be viewed as the graph of a function $y = y(x)$ (think about the circles). Again, C is determined by initial values (i.e., a point in the x-y plane) of the equation. You may be excited that we have ways to “fill” the plane with such curves. For example, the plane is filled by circles centered at the origin of different radii, and this is characterized by the differential equation

$$x + y\frac{dy}{dx} = 0.$$

Second, the method above should work for any equations of the form

$$p(x) + q(y)\frac{dy}{dx} = 0,$$

as long as you are able to find the integrals of p and q. Equations of this form are called *separable*, since one could view them as

$$p(x)dx = -q(y)dy,$$

where the variables are separated to each side of the equation, so that the both sides can be integrated independently.

Example. There are cases in which the equation does not appear separable, but becomes separable after a change of variables. For example, consider the equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}.$$

If we let $z = \frac{y}{x}$, then

$$\begin{aligned}\frac{dz}{dx} &= \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} \\ &= \frac{1}{x}\frac{dy}{dx} - \frac{z}{x},\end{aligned}$$

and

$$\frac{x^2 + y^2}{xy} = \frac{x}{y} + \frac{y}{x} = \frac{1}{z} + z.$$

Now the the function z satisfies

$$x\frac{dz}{dx} + z = \frac{dy}{dx} = \frac{1}{z} + z,$$

that is,

$$-\frac{1}{x} + z\frac{dz}{dx} = 0,$$

which is separable. More discussion on this example can be found in Ex. 30, Sec. 2.2.

2. Existence and Uniqueness

Going back to the list in the previous section, we see that the second equation does not make sense if our initial condition is set outside of the region $\{1 < x < 3\}$. Hence, before starting solving any equation, there is the issue of existence of solutions, given certain initial condition. You might think, as long as the initial condition is chosen so that the equation itself makes sense, then solution exists. Consider the equation:

$$y\frac{dy}{dx} = -x.$$

Remember that this equation characterizes the circles in the plane with radius $R \geq 0$. If we set $(x_0, y_0) = (0, 0)$ as the initial condition, evidently, there is no solution curve extending this initial value.

Apart from existence, a question to ask along with it is: If solutions exist for a given initial condition, are they unique? Again, the question is not trivial because of examples such as this:

$$\frac{dy}{dt} = y^{1/3}.$$

If we solve using the separation of variables, we'll obtain $y = \pm(\frac{2}{3}t + C)^{\frac{3}{2}}$ wherever it is defined. If we set the initial condition to be $y(0) = 0$, then $C = 0$ and $y = \pm(\frac{2}{3}t)^{\frac{3}{2}}$. This is already two distinct solutions! Moreover, by speculation, you may notice that there is another solution sharing the same initial condition, that is, $y = 0$. It turns out that the source of this non-uniqueness lies in the fact that $y^{1/3}$ has an infinite slope at $y = 0$.

Theorem(Existence and Uniqueness). For the first order initial value problem

$$\frac{dy}{dx} = g(x, y), \qquad y(x_0) = y_0,$$

if $g(x, y)$ and $\frac{\partial}{\partial y}g(x, y)$ are both continuous in some rectangle $(a, b) \times (c, d)$ containing the point (x_0, y_0), then the initial value problem has a local solution, i.e., there exists some open interval $(x_0 - \epsilon, x_0 + \epsilon) \subset (a, b)$ on which the solution to the initial value problem exists and is unique.

The Existence and Uniqueness theorem is useful in helping us determine the intervals on which solutions exist.

Example. Consider the following equation:

$$\frac{dy}{dx} = \frac{1}{1 - y}.$$

Because $\frac{1}{1-y}$ and $\frac{d}{dy}\frac{1}{1-y}$ are continuous except at $y = 1$, then the equation has unique solution near the initial values $y(x_0) = y_0 \neq 1$.

Note that the theorem only guarantees the existence of solution nearby the initial values, and one cannot expect the solution to be defined for all x.

Example. Consider the equation

$$\frac{dy}{dt} = 1 + y^2.$$

We could see that both $1 + y^2$ and $\frac{d}{dy}(1 + y^2) = 2y$ are continuous functions. Hence, for all initial values $y(x_0) = y_0$, solution exists. In fact, we could solve the equation by separation of variables, and it gives $y(x) = \tan(x + C)$. Clearly, once C is determined by initial values, say, $C = 0$ for example, the solution only exists within the interval $(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2})$ which contains x_0.

Questions. (1) For the differential equation whose solution curves are pieces of circles (see previous section), we see that it can be rewritten as

$$\frac{dy}{dx} = -\frac{x}{y}.$$

What does the existence and uniqueness theorem tell us if we set initial values $(x_0, y_0)(y_0 \neq 0)$. To what extent could we extend an interval on which solution exists?

(2) For the differential equation

$$\frac{dy}{dx} = y^{1/3},$$

we have seen that solutions are not unique given the initial condition $y(0) = 0$. By the Existence and Uniqueness theorem, we know that a unique solution locally exists if we set the initial condition to be $y(0) = y_0 \neq 0$. This time, to what extent can we extend the interval containing x_0 so that the solution remains unique?