LECTURE 19: SEPARATION OF VARIABLES, HEAT CONDUCTION IN A ROD

The idea of *separation of variables* is simple: in order to solve a partial differential equation in $u(x,t)$, we ask, is it possible to find a solution of the form

$$u(x,t) = X(x)T(t),$$

where X, T are functions of x and t, respectively. In this lecture, we are going to see how this idea is used in solving the equation of heat conduction in a rod.

1. The Heat Equation $\alpha^2 u_{xx} = u_t$

Think of a rod with the following data:

- *Length*: L;
- *Distribution of temperature along the rod*: $u(x,t)$, depending on time;
- *Area of cross section*: A, a small number so that the temperature is viewed as a constant when the x coordinate is fixed.

We also need three postulates:

- The rate of heat conduction crossing a unit area is proportional to $\frac{\partial u}{\partial \mathbf{n}}$, where $\mathbf{n}$ is the inward unit normal of the area. In our case, letting ΔH denote the heat transferred past (to the right) the cross section $x = x_0$, we have

$$\Delta H = -\lambda A \frac{\partial u}{\partial x}(x_0, t)\Delta t,$$

  for some constant $\lambda > 0$.
- Fixing a short segment in the rod, with volume $\Delta V = A \cdot \Delta x$, the increase in temperature times volume in this segment is proportional to the increase in heat in this segment.
- No heat transfer through the "horizontal" boundary of the rod.

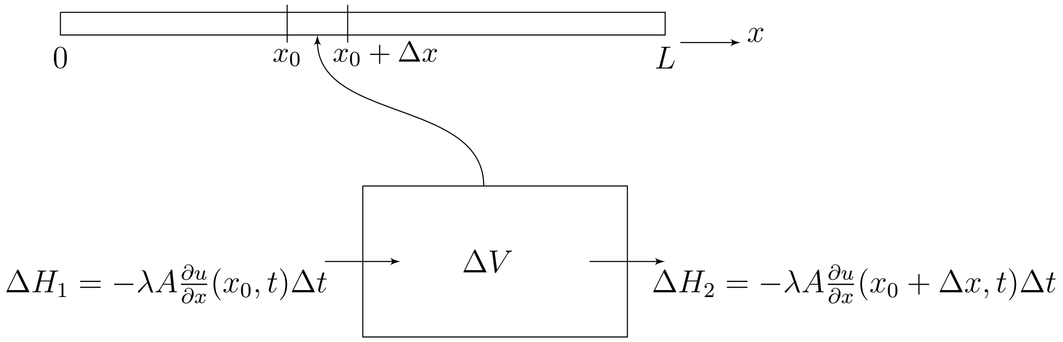

Figure 1. Heat Conduction in a Rod

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Now, focus on a segment in the rod shown in the figure above. Within Δt time, ΔV takes in heat from its left face by the amount ΔH_1 and loses heat from its right face by the amount ΔH_2. Hence, the heat in ΔV is increased by

$$\Delta H_1 - \Delta H_2 = \lambda A \Delta t \Big(\frac{\partial u}{\partial x}(x_0 + \Delta x, t) - \frac{\partial u}{\partial x}(x_0, t) \Big).$$

By the second postulate, this increase in heat is proportional to $\Delta u(x_0, t) \cdot \Delta V$, say,

$$\Delta H_1 - \Delta H_2 = \mu \Delta u(x_0, t) \cdot \Delta V = \mu \frac{\partial u}{\partial t}(x_0, t) \Delta t A \Delta x.$$

Therefore, we have the relation:

$$\lambda A \Delta t \Big(\frac{\partial u}{\partial x}(x_0 + \Delta x, t) - \frac{\partial u}{\partial x}(x_0, t) \Big) = \mu \frac{\partial u}{\partial t}(x_0, t) A \Delta t \Delta x,$$

i.e.,

$$\frac{\lambda}{\mu} \frac{1}{\Delta x} \Big(\frac{\partial u}{\partial x}(x_0 + \Delta x, t) - \frac{\partial u}{\partial x}(x_0, t) \Big) = \frac{\partial u}{\partial t}(x_0, t).$$

Passing to the limit $\Delta x \to 0$ gives

$$\frac{\lambda}{\mu} \frac{\partial^2 u}{\partial x^2}(x_0, t) = \frac{\partial u}{\partial t}(x_0, t).$$

Letting $\frac{\lambda}{\mu} = \alpha^2$, and noting that the equality above holds for all $0 < x_0 < L$, we have thus obtained

$$\alpha^2 u_{xx}(x, t) = u_t(x, t), \qquad (0 < x < L, t > 0),$$

the equation of heat conduction in a rod.

2. Solution of Initial-Boundary Value Problem

The equation established in the previous section characterizes the physical laws. In order to know exactly how the distribution of temperature in the rod evolves over time, we need more information. One of the simplest is, constant zero temperature at both ends of the rod together with an initial distribution of temperature inside the rod. This belongs to what's called an *initial-boundary value problem.* In symbols,

$$\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, t > 0, \\ u(0, t) = u(L, t) = 0, & t > 0, \\ u(x, 0) = f(x), & 0 \le x \le L. \end{cases}$$

We proceed to solve this initial-boundary value problem.

2.1. **Separation of Variables.** The first crucial step is called the *separation of variables.* Simply put, we are looking for solutions in the form

$$u(x, t) = X(x) T(t).$$

Plugging this in the equation, we obtain

$$\alpha^2 X''(x) T(t) = X(x) T(t),$$

which is,

$$\frac{X''}{X}(x) = \frac{T'}{\alpha^2 T}(t).$$

In this equality, the left hand side does not depend on t and the right hand side does not depend on x, hence they simply do not depend on the variables x, t at all, i.e., equal to some constant. So we let

$$\frac{X''}{X}(x) = \frac{T'}{\alpha^2 T}(t) = -\lambda,$$

for some constant λ. Or, written as two separate equations:

$$\begin{cases} X'' + \lambda X = 0, \\ T' + \alpha^2 \lambda T = 0. \end{cases}$$

We know how to find $X(x)$ and $T(t)$ since they are ODEs of the simplest type.

2.2. **Boundary Values.** Now we consider the boundary values. To start with, we would assume that the solution is not constantly zero, which is the case, as we could imagine, when the initial condition $u(x, 0) = f(x)$ is not constantly zero. With this assumption, the boundary values tell us

$$u(0, t) = X(0)T(t) = 0, \qquad u(L, t) = X(L)T(t) = 0,$$

for all $t > 0$. Hence,

$$X(0) = X(L) = 0.$$

If we focus on $X(x)$ at the moment, it is a nonzero solution of the two-point boundary value problem

$$\begin{cases} X'' + \lambda X = 0, \\ X(0) = X(L) = 0. \end{cases}$$

In looking for X, we are in fact looking for the eigenvalues and eigenfunctions of this boundary value problem. Note that the general solution of the equation

$$X'' + \lambda X = 0$$

are

$$X(x) = \begin{cases} c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x, & \lambda > 0, \\ c_1 x + c_2, & \lambda = 0, \\ c_1 e^{\sqrt{-\lambda} x} + c_2 e^{-\sqrt{-\lambda} x}, & \lambda < 0. \end{cases}$$

It is easy to see that nonzero solutions of the boundary value problem exist only when $\lambda > 0$ and $\sqrt{\lambda} L = n\pi$ $(n = \pm 1, \pm 2, ...)$, i.e.,

$$\lambda = \left(\frac{n\pi}{L}\right)^2, \qquad n = \pm 1, \pm 2, ...$$

Therefore, we have solutions (up to multiplication by a constant):

$$X_n(x) = \sin \frac{n\pi}{L} x,$$

and correspondingly,

$$T_n(t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$$

In sum, for each $n = 1, 2, 3, ...$, we have obtained the solution

$$u_n(x, t) = X(x)T(t) = \sin \frac{n\pi x}{L} \, e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t},$$

of the heat equation with the zero boundary value.

2.3. **Principle of Superposition.** We haven't considered the initial value $u(x,0) = f(x)$ yet. For the boundary value problem

$$\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0. \end{cases}$$

we have obtained a solution $u_n(x,t)$ for each $n = 1, 2, 3, ...$ And we do not know whether any of these solutions satisfies the initial condition.

Now, noting that this boundary value problem is linear and the boundary values are zero, we have that

$$\sum_{n=1}^{\infty} b_n u_n(x,t)$$

is a solution of the same boundary value problem for arbitrary constants $c_1, c_2, ...,$ as long as the infinite sum converges. This is called the *principle of superposition*. We ask: Is it possible to find the constants b_n so that

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t)$$

satisfies the initial condition? In other words, what are the b_n's such that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n X_n(x) T_n(0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}?$$

Aah! The b_n's, if exist, are just the Fourier coefficients when we expand $f(x)$ as a sine series! Therefore, we extend $f(x)$ oddly to the interval $[-L, L)$ and make it periodic with period $2L$. It follows that

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx,$$

and the series

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

converges to $f(x)$ at all points of continuity of $f(x)$. (If the initial value $f(x)$ is continuous, then the Fourier series simply converge to $f(x)$ for $0 < x < L$.)

To conclude, the solution of the initial-boundary value problem

$$\begin{cases} \alpha^2 u_{xx} = u_t, & 0 < x < L, t > 0, \\ u(0,t) = u(L,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 \le x \le L. \end{cases}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \, e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Example. Find the the distribution of temperature in a rod of length $L = 50$ with initial value $u(x,0) = f(x) = 20$. Temperature at both ends are set to be zero.

In this setting, $L = 50$ and

$$b_n = \frac{2}{50} \int_0^{50} 20 \sin \frac{n\pi x}{50} dx = \frac{40}{n\pi}(1 - (-1)^n).$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{40}{n\pi}(1 - (-1)^n) \sin \frac{n\pi x}{50}\ e^{-\frac{\alpha^2 n^2 \pi^2}{2500} t}.$$