LECTURE 16: TWO-POINT BOUNDARY VALUE PROBLEMS

Imagine that some player is juggling (vertically) a ball of unit mass. We are familiar that the motion of the ball is characterized by the Newton's law, a second order ordinary differential equation:

$$h'' = -g.$$

There are two question one could ask: one, what is the *trajectory* of the ball after it is launched at position h_0 with the velocity v_0; two, is it possible to receive the ball back two seconds after it is launched. It is easy to see that the first question corresponds to solving an initial value problem with initial conditions $h(0) = h_0, h'(0) = v_0$. For the second question, the conditions are $h(0) = h(2) = 0$. Since the values are fixed at the boundary of a time interval, we call it a *boundary value problem.* It is worth noting that while the trajectory of the ball always exists, given the initial position and velocity, it is not clear whether it is possible to obtain a solution for a boundary value problem. This is what we will discuss next and we are going to focus on second order equations.

Definition. The second order linear ordinary boundary value problem

$$y'' + p(x)y' + q(x)y = g(x), \qquad y(\alpha) = b_0,\ y(\beta) = b_1,\ \alpha \neq \beta$$

is said to be *homogeneous* if $g(x) = 0$ and $b_0 = b_1 = 0$.

For second order linear ODEs, in general, solving a boundary value problem is analogous to solving an initial value problem. The procedure is finding the general solution of the equation first, with certain coefficients to be determined. Then we determine those coefficients using the boundary values. For example, consider the boundary value problem

$$y'' + 3y = 0, \qquad y(0) = 1, y(\pi) = 0.$$

The general solution of the equation is

$$y(t) = c_1 \cos \sqrt{3}t + c_2 \sin \sqrt{3}t.$$

Using the boundary values, we obtain

$$\begin{aligned} c_1 &= 1, \\ c_1 \cos \sqrt{3}\pi + c_2 \sin \sqrt{3}\pi &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} c_1 &= 1, \\ c_2 &= -\cot \sqrt{3}\pi. \end{aligned}$$

Here, we obtained a unique solution of the boundary value problem.

Now, consider the boundary value problem

$$y'' + y = 0, \qquad y(0) = 1, y(\pi) = a,$$

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where a is some constant. Again, the general solution of the equation can be easily found to be

$$y(t) = c_1 \cos t + c_2 \sin t.$$

Taking in to account the boundary values,

$$c_1 = 1, \qquad -c_1 = a.$$

Two possibilities:

- $a = -1$. In this case, $c_1 = 1$ and c_2 can be any value. We have infinitely many solutions.
- $a \neq -1$. In this case, there are no solutions.

From the discussion above, we see that a boundary value problem may have unique, infinitely many or no solutions. Now we are going to explore a little more of this for equations of the form

$$y'' + \lambda y = 0, \qquad y(0) = 0, y(\pi) = 0.$$

Note that the zero solution is always a solution of this boundary value problem. In light of linear algebra, we call the values of λ such that nonzero solutions exist the *eigenvalues* of the boundary value problem. The corresponding nonzero solutions are called *eigenfunctions*. So, what are the eigenvalues and eigenfunctions of this boundary value problem? We discuss this in three cases:

- $\lambda = 0$. The equation becomes $y'' = 0$ and the general solutions are linear functions, i.e., $y(t) = c_1 + c_2 t$. Taking into account the boundary values, we have

  $$c_1 = 0, \qquad c_1 + c_2\pi = 0.$$

  Therefore, $c_1 = c_2 = 0$. The only solution is $y(t) = 0$. Hence, $\lambda = 0$ is not an eigenvalue.
- $\lambda < 0$. Let $\lambda = -\mu^2$, where $\mu > 0$ is a real number. The equation becomes $y'' - \mu^2 y = 0$ and the general solutions are

  $$y(t) = c_1 e^{\mu t} + c_2 e^{-\mu t}.$$

  By the boundary values, we have

  $$c_1 + c_2 = 0,$$
  $$c_1 e^{\mu\pi} + c_2 e^{-\mu\pi} = 0.$$

  One could write this system of linear equations in the matrix form $A\mathbf{c} = 0$ and see that, since $\mu > 0$, $\det A \neq 0$. Therefore, $c_1 = c_2 = 0$ and there are no negative eigenvalues of the given boundary value problem.
- $\lambda > 0$. Let $\lambda = \mu^2$ where $\mu > 0$. The equation becomes $y'' + \mu^2 y = 0$ and the general solutions are

  $$y(t) = c_1 \cos \mu t + c_2 \sin \mu t.$$

  Plugging in the boundary values, we have

  $$c_1 = 0,$$
  $$c_2 \sin \mu\pi = 0.$$

  Clearly, $\lambda = \mu^2$ is not an eigenvalue unless $\mu = k$ is an integer, i.e., $\lambda = 1, 4, 9, 16....$ If μ is an integer, then there are infinitely solutions, all in the form $c \sin \mu t$.