LECTURE 15: IMPULSE FUNCTIONS, CONVOLUTION INTEGRALS

We are still in the realm of constant coefficient ODEs. This time, we study two more types of input functions: (1) impulse functions; (2) functions that are expressed as a product.

1. The Impulse Function

Imagine a mass being placed at the origin and the mass has unit weight and takes zero space. In doing this, you've come up with what's called the *impulse function* or the *Dirac δ-function*. Formally, it is defined as

$$\delta(t) = 0, \qquad t \neq 0,$$
$$\int_{-\infty}^{\infty} \delta(t)dt = 1.$$

Note that the δ-function is not a function in the traditional sense, since its value at $t = 0$ is not defined. However, it turned out that various operations on regular functions can also be applied to this δ-function to obtain interesting results.

For example, by the definition of the δ-function, we have

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

To see this, note that $\delta(\tau) = 0$ for all $\tau < 0$ and that $\delta(\tau) = 0$ for all $\tau > 0$. In particular, for $t \geq 0$,

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \int_{-\infty}^{\infty} \delta(\tau)d\tau - \int_{\tau>t} \delta(\tau)d\tau = 1 - 0 = 1.$$

This seems to be telling us that the anti-derivative of the function $\delta(t)$ is the step function $u_0(t)$, in other words, $u_0'(t) = \delta(t)$. Right now, you may be exited, seeing that we might simply use the derivative property to calculate $\mathcal{L}\{\delta(t)\}$. However, this would not work properly, since, remember, we require the continuity of $f(t)$ in order for $\mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}$ to hold.

Exercise. Try calculating $\mathcal{L}\{\delta(t)\}$ as $\mathcal{L}\{u_0'(t)\}$. Compare this with the value of $\mathcal{L}\{\delta(t)\}$ which we'll obtain later in this lecture.

Besides the definition, another way to characterize the impulse function is viewing it as the limit of a sequence of functions:

$$d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau \leq t \leq \tau, \\ 0, & \text{elsewhere}, \end{cases}$$

as $\tau \to 0^+$. These functions are called the *top-hat* functions because of the shapes of their graphs. Note that the integral over the real line of these functions are always equal to 1, and for any $t \neq 0$, as τ gets small enough, we would have $d_\tau(t) = 0$. Now, it is easy to believe that the limit "function" is just $\delta(t)$.

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2. $\mathcal{L}\{\delta(t)\}$

An important property of the impulse function $\delta(t)$ is the following.

Proposition. If $f(t)$ is continuous in a neighborhood of $t = 0$, then

$$\int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0).$$

An informal way to see this is that we could view $f(t)$ as an "amplifier". Since $\delta(t)$ represents a mass of unit weight at the origin, it is amplified to a mass of weight $f(0)$. Therefore, the integral has to be $f(0)$.

Another approach is by calculating the integral

$$\int_{-\infty}^{\infty} d_\tau(t)f(t)dt = \frac{1}{2\tau}\int_{-\tau}^{\tau} f(t)dt.$$

By the continuity of $f(t)$ and the mean value theorem, there exists a $t^* \in [-\tau, \tau]$ such that $\frac{1}{2\tau}\int_{-\tau}^{\tau} f(t)dt = f(t^*)$. Now, let $\tau \to 0^+$. Correspondingly, $t^* \to 0$. Therefore, passing to the limit, we have

$$\lim_{\tau\to 0^+}\int_{-\infty}^{\infty} d_\tau(t)f(t)dt = \lim_{t^*\to 0} f(t^*) = f(0).$$

You may notice, still we need that the limit and the integral to be interchangeable in order to arrive at

$$\int_{-\infty}^{\infty} \delta(t)f(t)dt = f(0).$$

We are not getting into the details here. For now, just be aware that a gap exists in our argument.

Starting from the proposition, we could get the following corollaries:

Corollary 1. If $f(t)$ is continuous in a neighborhood of $t = t_0$, then

$$\int_{-\infty}^{\infty} \delta(t-t_0)f(t)dt = f(t_0).$$

The proof is simply by a change of variable: $\xi = t - t_0$ and then using the proposition above.

Corollary 2. For $t_0 > 0$, we have

$$\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}.$$

Proof. Since $t_0 > 0$, we have $\delta(t-t_0) = 0$ for $t \le 0$. Further more, since e^{-st} is continuous in t, by corollary 1,

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^{\infty} \delta(t-t_0)e^{-st}dt = \int_{-\infty}^{\infty} \delta(t-t_0)e^{-st}dt = e^{-st_0}.$$

Finally, for the consistency of applying the properties of the Laplace transform, we set

$$\mathcal{L}\{\delta(t)\} = 1,$$

according to the limit

$$\lim_{t_0 \to 0^+} \mathcal{L}\{\delta(t-t_0)\} = \lim_{t_0 \to 0^+} e^{-st_0} = 1.$$

3. Initial Value Problems with Impulse Forcing Functions

Given the Laplace transform of the impulse function, no doubt that we could solve differential equations with impulse forcing functions. We illustrate this with an example.

Example. Consider the initial value problem

$$2y'' + y' + 2y = \delta(t-5), \qquad y(0) = y'(0) = 0.$$

Routinely, apply the Laplace transform to both sides of this equation, obtaining

$$(2s^2 + s + 2)Y(s) = e^{-5s}.$$

Therefore,

$$Y(s) = e^{-5s} \frac{1}{2s^2 + s + 2}.$$

Again, if $h(t)$ satisfies

$$\mathcal{L}\{h(t)\} = \frac{1}{2s^2 + s + 2},$$

then, by the second shift theorem,

$$y(t) = u_5(t) h(t-5).$$

Now it suffices to find $h(t)$. In fact,

$$\frac{1}{2s^2 + s + 2} = \frac{1}{2} \frac{1}{(s+\frac{1}{4})^2 + \frac{15}{16}} = \frac{2}{\sqrt{15}} \frac{\sqrt{15}/4}{(s+\frac{1}{4})^2 + \frac{15}{16}}.$$

It is clear that

$$h(t) = \mathcal{L}^{-1}\{(2s^2 + s + 2)^{-1}\} = \frac{2}{\sqrt{15}} e^{-\frac{1}{4}t} \sin \frac{\sqrt{15}}{4} t.$$

Conclusion,

$$y(t) = u_5(t) \frac{2}{\sqrt{15}} e^{-\frac{1}{4}(t-5)} \sin \frac{\sqrt{15}}{4}(t-5).$$

Remark. Looking closer at the example, we could observe the following:

- Since the initial values are zero, the Laplace transform of the left hand side is simply the characteristic polynomial of the equation.
- The function $h(t)$ only depends on the left hand side of the equation, in other words, the characteristic polynomial of the equation. This determines qualitatively whether the solution is periodic/damping/oscilatting/diverging.
- The solution $y(t)$ is a shift of $h(t)$, depending on what the impulse function looks like.

In light of this, consider the equation

$$y'' + ay' + by = \delta(t-t_0), \qquad y(0) = y'(0) = 0, t_0 > 0$$

in general. We have the characteristic polynomial

$$p(s) = s^2 + as + b = (s - r_1)(s - r_2),$$

where r_1, r_2 are roots. Three possibilities:

- $r_1 \neq r_2$, real. In this case, $p(s)^{-1}$ must be of the form
$$\frac{A}{s-r_1} + \frac{B}{s-r_2}$$
for some constants A, B. Thus, $h(t)$ is the linear combination of the exponential functions e^{r_1}, e^{r_2} and $y(t)$ is the result of $h(t)$ being shifted to the right by t_0.
- $r_1 = r_2$. In this case,
$$p(s)^{-1} = \frac{1}{(s-r_1)^2}.$$
Thus, $h(t)$ is the function $te^{r_1 t}$ and $y(t) = u_{t_0}(t)(t-t_0)e^{r_1(t-t_0)}$.
- $r_1, r_2 = \lambda \pm i\mu$, $\mu \neq 0$. In this case,
$$p(s)^{-1} = \frac{1}{\mu}\frac{\mu}{(s-\lambda)^2+\mu^2}.$$
Thus,
$$h(t) = \mathcal{L}^{-1}\{p(s)^{-1}\} = \frac{1}{\mu}e^{\lambda t}\sin \mu t,$$
and
$$y(t) = u_{t_0}(t)h(t-t_0) = u_{t_0}(t)\frac{1}{\mu}e^{\lambda(t-t_0)}\sin \mu(t-t_0).$$

This completely characterizes the solution of initial value problems of the type
$$y'' + ay' + by = \delta(t-t_0), \qquad y(0) = y'(0) = 0, t_0 > 0.$$

4. Convolution Integrals

The definition of the *convolution* of two functions is straightforward.

Definition. Given $f(t), g(t)$ defined for $t \geq 0$, the *convolution* of f and g is defined as
$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau.$$

Properties.

- Commutative law: $f * g = g * f$.
Proof. By a change of variable $s = t - \tau$, we have
$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = -\int_t^0 f(s)g(t-s)ds$$
$$= \int_0^t f(s)g(t-s)ds = (f * g)(t).$$
- Distributive law: $f * (g_1 + g_2) = f * g_1 + f * g_2$.
Proof. This follows immediately from the linearity of the integral.

- Associative law: $(f * g) * h = f * (g * h)$.
  Proof. By definition,

$$\begin{aligned}((f * g) * h)(t) &= \int_0^t (f * g)(t-s)h(s)ds \\ &= \int_0^t \Big(\int_0^{t-s} f(t-s-\tau)g(\tau)d\tau\Big)h(s)ds \\ &= \int_0^t \int_0^{t-s} f(t-s-\tau)g(\tau)h(s)d\tau ds.\end{aligned}$$

  On the other hand,

$$\begin{aligned}(f * (g * h))(t) &= \int_0^t f(t-s)(g * h)(s)ds \\ &= \int_0^t f(t-s) \int_0^s g(s-\tau)h(\tau)d\tau ds \\ &= \int_0^t \int_0^s f(t-s)g(s-\tau)h(\tau)d\tau ds \\ (\tilde{s} = \tau, \tilde{\tau} = s - \tau) &= \int_0^t \int_0^{t-\tilde{s}} f(t-\tilde{\tau}-\tilde{s})g(\tilde{\tau})h(\tilde{s})d\tilde{\tau}d\tilde{s}.\end{aligned}$$

  Clearly, $(f * g) * h = f * (g * h)$.

The following theorem tells us that the inverse Laplace of a product of functions is equal to the convolution of the respective inverse Laplace transforms of those functions.

Theorem. If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$, where $F(s), G(s)$ are defined for $s > \alpha$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

Proof. It suffices to show that

$$\int_0^\infty (f * g)(t)e^{-st}dt = \int_0^\infty f(t)e^{-st}dt \int_0^\infty g(t)e^{-st}dt.$$

The right hand side of this equality can be rewritten as

$$\int_0^\infty \int_0^\infty f(t)g(\tau)e^{-s(t+\tau)}d\tau dt.$$

Now, let $x = t + \tau$ be a substitute for τ with t fixed, the expression above is just

$$\begin{aligned}\int_0^\infty \int_t^\infty f(t)g(x-t)e^{-sx}dxdt &= \int_0^\infty \Big(\int_0^t f(t)g(x-t)dt\Big)e^{-sx}dx \\ &= \int_0^\infty (g * f)(t)e^{-sx}dx,\end{aligned}$$

where the first equality comes from reversing the order of integration in the region $\{t \geq 0, x \geq t\} \subset \mathbb{R}^2$. Finally, note that $g * f = f * g$. This completes the proof.

Example. In a previous lecture, it took us a fair amount of effort to find $\mathcal{L}^{-1}\{(1+s^2)^{-2}\}$. Now, by the theorem, we know immediately that

$$\begin{aligned}\mathcal{L}^{-1}\{(1+s^2)^{-2}\} = \mathcal{L}^{-1}\{(1+s^2)^{-1}\} * \mathcal{L}^{-1}\{(1+s^2)^{-1}\} &= \sin t * \sin t \\ &= \int_0^t \sin(t-\tau)\sin(\tau) d\tau \\ &= \int_0^t \frac{1}{2}[\cos(t-2\tau) - \cos t] d\tau \\ &= \frac{1}{4}[-\sin(t-2\tau) - 2\tau\cos t]\Big|_{\tau=0}^{t} \\ &= \frac{1}{2}(\sin t - t\cos t).\end{aligned}$$

Exercise. Consider $F(s)$ as $F(s)\cdot 1$ and note that $\mathcal{L}^{-1}\{1\} = \delta(t)$. Now apply the convolution theorem to finding $\mathcal{L}^{-1}\{F(s)\cdot 1\}$, what do you get?
Ans. $f(t) = \delta(t) * f(t)$. This means, the impulse function is playing the role of a "multiplicative identity" in the sense of convolution.

Example. Using convolution, solve the initial value problem

$$y'' + 2y' + 2y = g(t), \qquad y(0) = y'(0) = 0.$$

Applying $\mathcal{L}$ on both sides of this equation gives

$$(s^2 + 2s + 2)Y(s) = G(s).$$

Thus,

$$Y(s) = \frac{G(s)}{s^2 + 2s + 2}.$$

Therefore,

$$\begin{aligned}y(t) &= g(t) * \mathcal{L}^{-1}\{(s^2+2s+2)^{-1}\} = G(s) * \mathcal{L}^{-1}\{((s+1)^2+1)^{-1}\} \\ &= g(t) * (e^{-t}\sin t).\end{aligned}$$

Note that we obtained $e^{-t}\sin t$ only from the left hand side, i.e., the "system". And the output is simply the convolution of the input $g(t)$ and a function intrinsic to the system. Of course, this kind of result is neither restricted to this particular set of constant coefficients, nor to this particular set of initial values. As an exercise, you may try to convince yourself of this. And the result should generalize the remark at the end of section 3.