LECTURE 14: STEP FUNCTIONS, DISCONTINUOUS INPUTS

One may have noticed that in the last lecture, the equations which we applied the Laplace transform to solve are familiar ones: second order with constant coefficients. We knew how to solve this kind of equations, using either undetermined coefficients or variation of parameters. Thus, it is natural to ask, what is the advantage of using the Laplace transform?

Given a constant coefficient linear ordinary differential equation of the form

$$y^{(n)}(t) + c_{n-1}y^{(n-1)}(t) + ... + c_1 y'(t) + c_0 y(t) = g(t),$$

an engineering-flavored way of viewing it is taking $g(t)$ as an input, the left hand side of the equation as a "black-box" of system, and the solutions of the equation as outputs. In fact, this point of view leads us to a different context, in which one starts to care about the specifics of the inputs. For instance, the inputs could be discontinuous. In this case, solving the differential equation using traditional methods may be tedious as one will need to solve the equation piece-by-piece on various intervals where the input function is continuous. As we will see in a minute, this kind of repeated effort is unnecessary if we take the Laplace transform approach.

1. Step Functions

Recall the Heaviside's step function

$$u_c(t) = \begin{cases} 1, & t \geq c, \\ 0, & 0 \leq t < c. \end{cases}$$

Previously we have shown that

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s),$$

for any appropriate function $f(t)$. As an exercise, you can convince yourself that the function $u_c(t)f(t-c)$ is the result of shifting the function $f(t)$ with $t \geq 0$ to the right by c. So it makes sense to call this result the *second shift theorem*.

In practice, we could use the step functions to come up with more functions that are piecewise continuous. For example, the function

$$f(t) = u_1(t) - u_2(t),$$

is simply the function whose value is one on $[1,2]$ and zero everywhere else. Conversely, given a function which is defined piecewise, we could use the step functions to rewrite it in a more compact form. For example, let

$$g(t) = \begin{cases} 1, & 0 \leq t < 2, \\ 4, & 2 \leq t < 5, \\ -3, & t \geq 5. \end{cases}$$

Using step functions, $g(t)$ can be rewritten as:

$$g(t) = 1 + 3u_2(t) - 7u_5(t), \quad t \geq 0.$$

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2. Differential Equations with Discontinuous Inputs

Solving constant coefficient ODEs with discontinuous inputs using the Laplace transform follows the same procedure as we have seen before. The only difference is that we need to turn the discontinuous inputs into a form to which the second shift theorem is applicable, and, in the inverse Laplace transform step, applications of the second shift theorem in the opposite direction may be needed. We use an example to illustrate this procedure.

Example. Solve the initial value problem

$$2y'' + y' + 2y = g(t), \qquad y(0) = y'(0) = 0,$$

where

$$g(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & \text{elsewhere.} \end{cases}$$

The first step is, of course, applying the Laplace transform on the both sides. In particular, since we have the zero initial values, the Laplace transform of the left hand side of this equation is quite simple.

$$\mathcal{L}\{2y'' + y' + 2y\} = (2s^2 + s + 2)Y(s).$$

Before applying $\mathcal{L}$ to the right hand side of the equation, let us rewrite $g(t)$ as

$$g(t) = u_5(t) - u_{20}(t).$$

Thus, by the second shift theorem and that $\mathcal{L}\{1\} = s^{-1}$, we have

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{u_5(t) - u_{20}(t)\} = (e^{-5s} - e^{-20s})\frac{1}{s}.$$

Equating the Laplace transforms on both sides of the equation gives

$$Y(s) = (e^{-5s} - e^{-20s})\frac{1}{s(2s^2 + s + 2)}.$$

Now, one could notice that if $h(t)$ is the inverse Laplace transform of $(s(2s^2 + s + 2))^{-1}$, then the inverse Laplace transform of $Y(s)$ is simply

$$y(t) = u_5(t)h(t-5) + u_{20}(t)h(t-20),$$

by the second shift theorem. So all we need to do is to find $h(t)$.

Let's use the partial fraction decomposition to turn $(s(2s^2 + s + 2))^{-1}$ into the sum of simpler terms: Supposing

$$\frac{A}{s} + \frac{Bs + C}{2s^2 + s + 2} = \frac{(2A + B)s^2 + (A + C)s + 2A}{s(2s^2 + s + 2)} = \frac{1}{s(2s^2 + s + 2)},$$

it follows that

$$2A + B = 0, \qquad A + C = 0, \qquad 2A = 1.$$

Thus,

$$A = \frac{1}{2}, \qquad , B = -1, \qquad C = -\frac{1}{2},$$

and

$$\frac{1}{s(2s^2 + s + 2)} = \frac{1}{2s} - \frac{1}{2}\frac{s + \frac{1}{2}}{s^2 + \frac{1}{2}s + 1} = \frac{1}{2s} - \frac{1}{2}\frac{(s + \frac{1}{4}) + \frac{1}{4}}{(s + \frac{1}{4})^2 + \frac{15}{16}}.$$

Now it is not hard to see that

$$\begin{aligned} h(t) &= \frac{1}{2} - \frac{1}{2}e^{-\frac{1}{4}t}\cos\frac{\sqrt{15}}{4}t - \frac{1}{2\sqrt{15}}e^{-\frac{1}{4}t}\sin\frac{\sqrt{15}}{4}t \\ &= \frac{1}{2} - \frac{1}{2}e^{-\frac{1}{4}t}\Big(\cos\frac{\sqrt{15}}{4}t + \frac{1}{\sqrt{15}}\sin\frac{\sqrt{15}}{4}t\Big). \end{aligned}$$

Finally, plugging this $h(t)$ in

$$y(t) = u_5(t)h(t-5) + u_{20}(t)h(t-20)$$

gives the solution.