LECTURE 12: LAPLACE TRANSFORM

1. Definition and Questions

The definition of the Laplace transform could hardly be simpler: For an appropriate function $f(t)$, the *Laplace transform* of $f(t)$ is a function $F(s)$ which is equal to

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt.$$

Naturally, just by looking at this expression, one might ask:

- What are the domain and range of this transformation?
- Is the transformation one-to-one? That is, does $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$ imply $f(t)=g(t)$?
- Is the transformation linear?
- What does $\int_0^\infty$ mean?
- How can this transformation be applied?

So, what are the domain and range of the Laplace transformation? To be more explicit, we are really asking, for which functions $f(t)$, and for which values of s is the integral

$$\int_0^\infty f(t)e^{-st}dt$$

well-defined? In fact, this improper integral is defined as the limit

$$\lim_{N\to\infty} \int_0^N f(t)e^{-st}dt.$$

Thus we are asking, for which functions $f(t)$ does this limit exist? Before a characterization of such functions, let's see some examples of the Laplace transform.

2. Examples

2.1. $\mathcal{L}\{1\}$. By the definition of the Laplace transform,

$$\begin{aligned}
\mathcal{L}\{1\} &= \lim_{N\to\infty} \int_0^N 1 \cdot e^{-st}dt \\
&= \lim_{N\to\infty} \Big(-\frac{1}{s}e^{-st}\Big|_{t=0}^{N}\Big) \\
&= \lim_{N\to\infty} \Big(-\frac{1}{s}e^{-sN} + \frac{1}{s}\Big).
\end{aligned}$$

This limit converges to s^{-1} for $s > 0$. Hence,

$$\mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0.$$

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2.2. $\mathcal{L}\{t\}$. By the definition,

$$\begin{aligned}\mathcal{L}\{t\} &= \lim_{N\to\infty}\int_0^N te^{-st}dt \\ &= \lim_{N\to\infty} -\frac{1}{s}\left[te^{-st}\Big|_{t=0}^{N} - \int_0^N e^{-st}dt\right] \\ &= -\frac{1}{s}\lim_{N\to\infty}(Ne^{-sN}) + \frac{1}{s}\mathcal{L}\{1\}\end{aligned}$$

For $s>0$, the limit exists and we have

$$\mathcal{L}\{t\} = \frac{1}{s^2}, \qquad s>0.$$

2.3. $\mathcal{L}\{t^n\}$. Again,

$$\begin{aligned}\mathcal{L}\{t^n\} &= \lim_{N\to\infty}\int_0^N t^n e^{-st}dt \\ &= \lim_{N\to\infty} -\frac{1}{s}\left[t^n e^{-st}\Big|_{t=0}^{N} - \int_0^N nt^{n-1}e^{-st}dt\right] \\ &= -\frac{1}{s}\lim_{N\to\infty}(N^n e^{-sN}) + \frac{n}{s}\mathcal{L}\{t^{n-1}\}.\end{aligned}$$

For $s>0$, by L'Hôpital's rule, we have $\lim_{N\to\infty}(N^n e^{-sN}) = 0$. Therefore,

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\}, \qquad s>0.$$

By induction,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^n}\mathcal{L}\{1\} = \frac{n!}{s^{n+1}}, \qquad s>0.$$

2.4. $\mathcal{L}\{\cos t\}$. By definition,

$$\mathcal{L}\{\cos t\} = \lim_{N\to\infty}\int_0^N \cos t\ e^{-st}dt$$

Now we need to evaluate the integral $\int \cos t\ e^{-st}dt$. By a trick introduced before, we could calculate the derivatives of $\cos t\ e^{-st}, \sin t\ e^{-st}$ with respect to t:

$$(\cos t\ e^{-st})' = -\sin t\ e^{-st} - s\cos t\ e^{-st}, \tag{1}$$

$$(\sin t\ e^{-st})' = \cos t\ e^{-st} - s\sin t\ e^{-st}. \tag{2}$$

So, (2) $- s\cdot$ (1) gives

$$(\sin t\ e^{-st} - s\cos t\ e^{-st})' = (1+s^2)\cos t\ e^{-st}.$$

Therefore,

$$\begin{aligned}\int_0^N \cos t\ e^{-st}dt &= \frac{1}{1+s^2}(\sin t - s\cos t)e^{-st}\Big|_{t=0}^{N} \\ &= \frac{e^{-sN}}{1+s^2}(\sin N - s\cos N) + \frac{s}{1+s^2}.\end{aligned}$$

For $s > 0$, taking $N \to \infty$ gives

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}, \qquad s > 0.$$

Remark. We will see a simpler way to find the Laplace transform of $\cos at$ and $\sin at$ once we have the *first shift theorem*.

2.5. $\mathcal{L}\{u_c(t)\}$. Here

$$u_c(t) = \begin{cases} 0, & t \in [0,c) \\ 1, & t \in [c,\infty) \end{cases}$$

is called the *Heaviside's step function*, where c is a non-negative constant.

Again, we are facing the integral

$$\int_0^N u_c(t)e^{-st}dt.$$

For N begin sufficiently large, this integral consists of two parts: integration on $[0,c)$ where $u_c(t) = 0$ and integration on $[c,N]$ where $u_c(t) = 1$. Therefore,

$$\mathcal{L}\{u_c(t)\} = \lim_{N\to\infty} \int_0^N u_c(t)e^{-st}dt = \lim_{N\to\infty} \int_c^N e^{-st}dt = \frac{1}{s}e^{-sc}, \qquad s > 0.$$

3. The "Appropriate" functions

To which functions can one apply the Laplace transform?

First of all, we need the integral $\int_0^N f(t)dt$ to make sense, i.e., $f(t)$ is integrable on the intervals $[0,N]$. The function $f(t)$ being continuous certainly guarantees this. However, we could also allow a finitely number of discontinuities, say $0 < x_1 < x_2 < ... < x_k < N$, so the integral is the sum of the integrals of $f(t)$ on each of the intervals $[0,x_1),[x_1,x_2),...,[x_k,N]$ where $f(t)$ is continuous. Such functions are called *piecewise continuous* on the interval $[0,N]$. A function is said to be *piecewise continuous on* $[0,\infty)$ if it is piecewise continuous on any $[0,N]$, $N > 0$.

Secondly, we need the limit $\lim_{N\to\infty}\int_0^N f(t)e^{-st}dt$ to exist. Intuitively, fixing s, this is saying that the signed area of the region enclosed by the graph of $f(t)e^{-st}$ and the x-axis is finite. One implication is that $f(t)$ should not increase too fast. For instance, for $s = 1$ and $f(t) = e^t$, we have $f(t)e^{-st} = 1$ and the limit of the integral does not exist. Also note that for the same $f(t)$, the limit of the integral exists for $s > 1$. More generally, if $f(t) = Me^{\alpha t}$ for some $M, \alpha > 0$, then the limit

$$\lim_{N\to\infty} \int_0^N Me^{\alpha t}e^{-st}dt = \lim_{N\to\infty} \int_0^N Me^{(\alpha-s)t}dt = \lim_{N\to\infty} \frac{M}{\alpha - s}(e^{(\alpha-s)N} - 1)$$

exists for all $s > \alpha$. What is exciting about this fact is that it actually brings a lot "appropriate functions" into our sight, based on the comparison theorem below.

Theorem. If $f(x)$ and $g(x)$ are piecewise continuous functions defined on $[0,\infty)$ and $|f(x)| \le g(x)$, then the convergence of $\int_0^\infty g(x)dx$ implies the convergence of $\int_0^\infty f(x)dx$.

The following corollary of the theorem tells us which functions are "appropriate" to be put in the Laplace transform.

Corollary. Based on the theorem, we can say that if (a) $f(t)$ *is piecewise continuous on* $[0, N]$ *for any* $N > 0$ and (b) *it is of the exponential order*, then $\int_0^\infty f(t)e^{-st}dt$ converges on the interval (α, ∞) for some $\alpha > 0$.

Note: A function $f(t)$ is said to be of the *exponential order* if there exist constants $M, \alpha > 0$ such that $|f(t)| \le Me^{\alpha t}$ on $[0, \infty)$.

From now on, we confine ourselves to the category of functions which satisfy the conditions in the corollary above and call these functions "appropriate". As an exercise, you can check that all the $f(t)$'s in the previous section are appropriate.

4. Properties of the Laplace Transform

4.1. **Linearity.** The Laplace transform is linear. In fact, for $f(t), g(t)$ appropriate functions and $\mathcal{L}\{f(t)\} = F(s)$, $\mathcal{L}\{g(t)\} = G(s)$, both defined for $s \in (\alpha, \infty)$, and any constants a, b, we have that

$$\begin{aligned}\mathcal{L}\{af(t) + bg(t)\} &= \int_0^\infty (af(t) + bg(t))e^{-st}dt \\ &= a\int_0^\infty f(t)e^{-st}dt + b\int_0^\infty g(t)e^{-st}dt \\ &= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}.\end{aligned}$$

is defined for $s \in (\alpha, \infty)$. This proves the linearity.

4.2. **First shift theorem.** If $f(t)$ is appropriate and $F(s) = \mathcal{L}\{f(t)\}$ is defined for $s \in (\alpha, \infty)$, then

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$$

is defined for $s \in (\alpha + a, \infty)$.

Proof. By definition,

$$\begin{aligned}\mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{at}f(t)e^{-st}dt \\ &= \int_0^\infty f(t)e^{-(s-a)t}dt \\ &= F(s-a).\end{aligned}$$

Since we know that $F(s)$ is defined for $s \in (\alpha, \infty)$, $F(s-a)$ is defined for $s \in (\alpha+a, \infty)$.

Remark. The first shift theorem works for complex-valued a. For instance, letting $a = i$, we have

$$\mathcal{L}\{e^{at} \cdot 1\} = \frac{1}{s-a} = \frac{s+i}{s^2+1}.$$

On the other hand, by linearity,

$$\mathcal{L}\{e^{it}\} = \mathcal{L}\{\cos t\} + i\mathcal{L}\{\sin t\}.$$

Hence, comparing the real and imaginary parts, we have

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}, \qquad \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}.$$

4.3. **Second shift theorem.** If $f(t)$ is appropriate and $F(s) = \mathcal{L}\{f(t)\}$ is defined for $s \in (\alpha, 0)$, then

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \qquad s > \alpha.$$

Proof. First, you may want to convince yourself that the function $u_c(t)f(t-c)$ is indeed the result of shifting $f(t)$ to the right by c.

By definition,

$$\begin{aligned}
\mathcal{L}\{u_c(t)f(t-c)\} &= \int_0^\infty u_c(t)f(t-c)e^{-st}dt \\
&= \int_c^\infty f(t-c)e^{-st}dt \\
&= \int_0^\infty f(\xi)e^{-s(\xi+c)d\xi} \qquad (\xi = t-c) \\
&= e^{-sc}\int_0^\infty f(\xi)e^{-s\xi}d\xi \\
&= e^{-sc}F(s), \qquad s > \alpha.
\end{aligned}$$

4.4. **Relation to the derivatives.**

4.4.1. $\mathcal{L}\{f'(t)\}$. Before stating the conditions on $f(t)$, let us calculate formally the Laplace transform of $f'(t)$:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt.$$

Now, using integration by parts (formally), we obtain

$$\begin{aligned}
\mathcal{L}\{f'(t)\} &= f(t)e^{-st}\Big|_{t=0}^{\infty} - (-s)\int_0^\infty f(t)e^{-st}dt \\
&= -f(0) + s\mathcal{L}\{f(t)\}.
\end{aligned}$$

Back to the question: what are the conditions on the function $f(t)$? First, the result above involves the Laplace transform of $f(t)$ and $f'(t)$, so it is reasonable to have both $f(t), f'(t)$ as being appropriate, i.e., piecewise continuous and of the exponential order. Second, by the fundamental theorem of calculus, the integration by parts only applies to functions that are continuous. You may wonder, isn't $f(t)$ continuous if $f'(t)$ exists? Remember, in our context, finitely many discontinuities can exist and $f'(t)$ may only exist piece-wisely. So, we enforce another condition, that is, $f(t)$ is continuous. To summarize, we state this as a proposition:

Proposition. If $f(t)$ is continuous and $f(t), f'(t)$ are appropriate, i.e., piecewise continuous and of the exponential order, then

$$\mathcal{L}\{f'(t)\} = -f(0) + s\mathcal{L}\{f(t)\}, \qquad s > \alpha$$

for some $\alpha > 0$.

Remark. In fact, α can be chosen using the fact that $f(t), f'(t)$ are of the exponential order. Do you see why?

4.4.2. $\mathcal{L}\{f^{(n)}(t)\}$. The previous derivative property can be applied successively and the result generalized to higher derivatives.

Proposition. If $f(t), f'(t), ..., f^{(n-1)}(t)$ are continuous and $f(t), ..., f^{(n)}(t)$ are appropriate, then

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - ... - f^{(n-1)}(0), \qquad s > \alpha$$

for some $\alpha > 0$.

Note, in particular, that when $n = 2$, this is just

$$\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - sf(0) - f'(0), \qquad s > \alpha.$$

4.4.3. $\mathcal{L}\{tf(t)\}$. Let us calculate $\mathcal{L}\{tf(t)\}$ formally by the definition

$$\begin{aligned}\mathcal{L}\{tf(t)\} &= \int_0^\infty tf(t)e^{-st}dt \\ &= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}dt \\ &= -\frac{d}{ds}\mathcal{L}\{f(t)\} \\ &= -\frac{d}{ds}F(s).\end{aligned}$$

Here we have implicitly interchanged the order of integral and derivative. Particularly, assuming the integral to be absolutely convergent ensures this. Of course, the result above can be extended to

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}.$$

As an exercise, you can prove this formula, then use it to find $\mathcal{L}\{t^n\}$.