LECTURE 1: INTRODUCTION, LINEAR EQUATIONS, INTEGRATING FACTORS

1. Introduction, Linear Equations

The subject of this course is differential equations, i.e., equations that involve the derivatives of the unknowns. As you may recall from calculus, perhaps one of the simplest example is the equation modeling population growth of organisms (with no environmental capacity constraint):

$$\frac{dp}{dt} = rp,$$

where p stands for the population and r is some constant. Note that the derivative in this equation is with respect to the time variable t. It is a convention to use $\dot{p}$, $\ddot{p}$... to denote the first, second,... derivatives of p with respect to t. So our equation above can also be written as

$$\dot{p} = rp.$$

If you remember Math216, here is an equation characterizing the motion of a (damped) harmonic oscillator:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t),$$

where m stands for mass, b for friction, k for Hooke's constant, and $f(t)$ for the external force/input. When $f(t) \equiv 0$, the equation is called homogeneous. We studied this type of equations in Math216.

It is an easy notice that the two equations above share the property that they both involve derivatives with respect to *only one* free variable, in this case, t. Differential equations having this property are called *ordinary differential equations* (ODEs). Further comparing these equations, we observe that one involves at most the first derivative while the other involves at most the second derivative of t. The number of the highest derivative taken in a differential equation is called the *order* of the differential equation.

Consider the equation modeling the heat transfer in a line:

$$\dot{u} = \frac{\partial^2 u}{\partial x^2}.$$

This equation differs from the previous two in that it involves partial derivatives of the unknown function $u = u(x,t)$. Equations that involve partial derivatives are called *partial differential equations* (PDEs). Note that the definition above of the *order* of an equation extends to the category of partial differential equations. For example, the order of the heat equation above is 2, and the order of the equation

$$\frac{\partial^5 u}{\partial x^2 \partial y^2 \partial z} = u$$

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is 5.

If we look at the four equations given above, a less-so-obvious feature that they share is that these equations are *linear*, in the sense that if we rewrite them in the form

$$f(t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}, ...) = 0,$$

where I've used u to denote all the unknown functions, f, viewed as a function of all the possible derivatives of u (including u itself), is a linear function. For example, in the population model, we could set

$$f(t, p, \dot{p}) = \dot{p} - rp,$$

which is a linear function in p and $\dot{p}$. In the Heat equation model, we could set

$$f(t, x, u, \dot{u}, \frac{\partial u}{\partial x}, \ddot{u}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial t}) = \dot{u} - \frac{\partial^2 u}{\partial x^2},$$

which is also linear. Symbolically, the expression

$$f(t, x, p, q, r) = (t + x^2)p + (t^2 \sin x)q + \Big(\frac{1}{t} \ln x\Big)r,$$

is linear in p, q, r but not linear in p, q, r, x nor in p, q, r, t. Differential equations that are not linear are called *nonlinear.*

Our course treats mainly the linear equations, for which the theory of analysis is well-established. However, one should note that, in many instances of modeling real life, linear equations are too good to be true. For instance, the population model above has exponential solutions, which might be the case when the population is small compared with the resource in the environment required for reproduction. However, when p gets large, one has to consider what's called the environmental capacity. In light of this, one could adjust the equation to be

$$\dot{p} = p(1 - \frac{p}{R}),$$

where R is the environmental capacity (Do you see why?). By the definition, this equation is clearly nonlinear. In fact, despite the hardness of finding explicit formulas for solutions of nonlinear equations, we could for this equation. As an exercise, you could choose $R = 1$, divide the the equation by its right hand side, and try to integrate the expressions you get.

At this point, hopefully you are convinced that there can be innumerably many differential equations out there, and we cannot look at them and try to solve one by one. Instead, just like what we observed in the equations above, we ask the question, can we classify the equations and develop theories to solve certain classes of equations? More subtly, given a particular equation, to what extent and how could we zoom out from it, so that a theory can be established? So far, we could guess, maybe the *order* of an equation matters, maybe *homogeneity*, *linear or nonlinear*, number of free variables, or maybe a combination of these features. This is exactly what we are going to do next. We start with the simplest possible of the combinations of the "features": linear first order ordinary differential equations.

2. Integrating Factors

In this section, we introduce the method of *integrating factors* which solves linear first order ordinary differential equations. First of all, what do these equations look like? Let's think. We have "first order" and "ordinary", thus the equation could only possibly involve $x, y, \frac{dy}{dx}$ and can be written as

$$f(x, y, \frac{dy}{dx}) = 0.$$

Furthermore, we have "linear", which renders f in the form

$$f(x, y, \frac{dy}{dx}) = s(x)\frac{dy}{dx} + q(x)y + r(x).$$

Thus our equation always takes the form

$$s(x)\frac{dy}{dx} + q(x)y + r(x) = 0.$$

In the context, we assume $p(x)$ to be nonzero in our domain of interest and the equation simplifies to

$$\frac{dy}{dx} + p(x)y = g(x),$$

for some smooth functions $p(x)$ and $g(x)$.

Now comes the task of solving these equations. In a particular instance when $p(x) \equiv 0$, the equation is simply

$$\frac{dy}{dx} = g(x),$$

and the solution is, as you may recall from calculus,

$$y = \int g(x)dx + C.$$

What if $p(x) \neq 0$? The **method of integrating factors** suggests the following approach: *If there exists a function $\mu(x)$ so that whenever $y(x)$ solves*

$$\frac{dy}{dx} + p(x)y = g(x),$$

$z(x) = \mu(x)y(x)$ solves some equation

$$\frac{dz}{dx} = \tilde{g}(x).$$

since the latter equation is readily solvable by integration, then dividing its solution by $\mu(x)$ yields $y(x)$. Such a $\mu(x)$ is called an integrating factor.

Looks like a promising idea? Question is, can we always find $\mu(x)$? Note that our goal is to find a function μ, such that in the equation that $z = \mu y$ satisfies, the coefficient of

z equals to zero. Written in formulas, we have that $z = \mu y$ satisfies

$$\begin{aligned}\frac{dz}{dx} &= \frac{d}{dx}(\mu y)\\ &= \frac{d\mu}{dx}y + \mu\frac{dy}{dx}\\ &= \frac{d\mu}{dx}y + \mu g(x) - \mu p(x)y\\ &= \Big(\frac{d\mu}{dx} - \mu p(x)\Big)y + \mu g(x)\\ &= \frac{1}{\mu}\Big(\frac{d\mu}{dx} - \mu p(x)\Big)z + \mu g(x).\end{aligned}$$

So if we set

$$\frac{d\mu}{dx} - \mu p(x) = 0,$$

solving this equation gives

$$\mu(x) = e^{\int p(t)dt},$$

then

$$\frac{dz}{dx} = \mu(x)g(x),$$

and immediately

$$z(x) = \int \mu(x)g(x)dx + C = \int e^{\int p(x)dx}g(x)dx + C.$$

Finally,

$$y(x) = \frac{z(x)}{\mu(x)} = e^{-\int p(x)dx}\Big(\int e^{\int p(x)dx}g(x)dx + C\Big).$$

Are we done? What about the unknown parameter C? You may recall that such C can be determined by initial conditions, such as $y(3) = 4$. The theory completes its job here. Ready for examples.

Example. Solve the initial value problem

$$y' - y = 1 + 2\cos x, \quad y(0) = y_0.$$

How would you set the values of y_0 so that the solution remains finite when $t \to \infty$?

Solution. Corresponding to the method of integrating factors, here $p(x) = -1$ and $g(x) = 1 + 2\cos x$. Thus the solution is

$$\begin{aligned}y(x) &= e^x(\int e^{-x}(1 + 2\cos x)dx + C)\\ &= e^x(-e^{-x} + e^{-x}(\sin x - \cos x) + C)\\ &= -1 + (\sin x - \cos x) + Ce^x.\end{aligned}$$

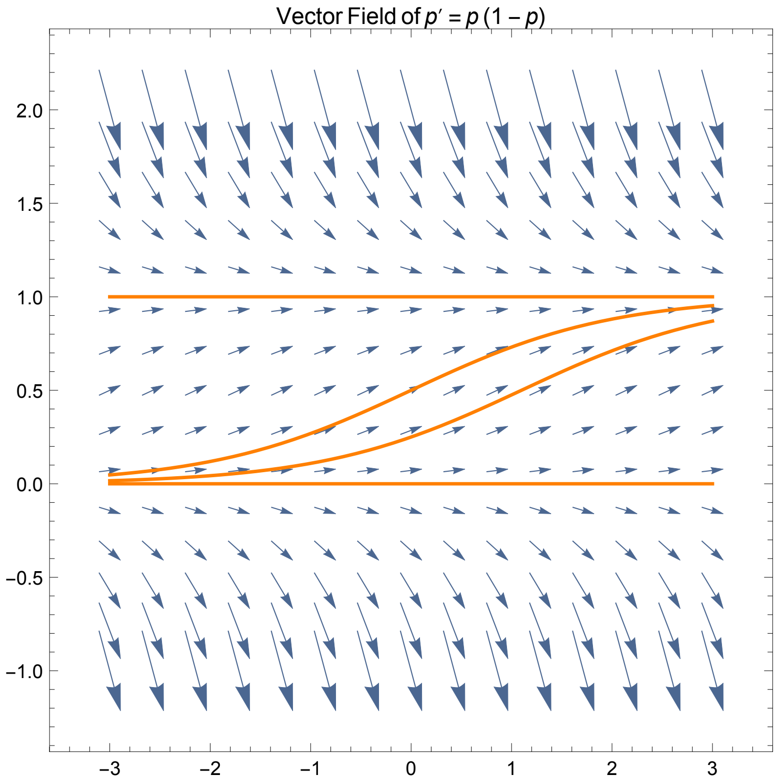
For the initial condition $y(0) = y_0$, we know that $C = y_0 + 2$. Hence, the solution remains finite as $t \to \infty$ if and only if $y_0 = -2$.

3. Direction/Vector Fields

Here, we briefly use the equation

$$\dot{p} = p(1-p)$$

as an example to explain the direction field (aka. slope field) associated to an ordinary differential equation and why it may be helpful in understanding the solutions. First notice that in the $t-p$ plane, a solution to the equation above is (a section of) a curve parametrized as $(t, p(t))$. By the differential equation, we see that the tangent segments at each point along this curve is parallel to the vector $(1, \dot{p}) = (1, p(t)(1-p(t)))$. In other words, if the $t-p$ plane is equipped with the vectors $(1, p(1-p))$ at each point, then a solution curve is simply one that is everywhere tangent to the vectors. See the diagram below:



As you can see, this equation has two constant solutions $p(t) = 0$ and $p(t) = 1$, along whose graph the vector fields are horizontal. Constant solutions such as these are called equilibria of the differential system. In our case, $p = 0$ is called a *stable equilibrium* in the sense that a small perturbation yields a solution which tends to $p = 1$ as t tends to infinity. For a similar reason, you can see that the equilibrium $p = 0$ is *unstable*. Now, do you see why, in this setting, $R = 1$ can be understood as the environmental capacity?