PRACTICE EXAM II SOLUTIONS

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. You have **75 minutes** to answer all the questions. *Good Luck* !

Laplace Transforms:

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{\sin at\} = \frac{a}{a^2+s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{\cos at\} = \frac{s}{a^2+s^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \qquad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s) \qquad \mathcal{L}\{f'(t)\} = -f(0) + sF(s)$$

$$\mathcal{L}\{\delta(t)\} = 1 \qquad \mathcal{L}\{(f*g)(t)\} = F(s)G(s)$$

Question	Max. Points	Score
1	15	
2	20	
3	25	
4	20	
5	10	
6	10	
Total	100	

1. (15 points) Which of the following functions are *appropriate* for the Laplace transform. For those that are appropriate, find the Laplace transform.

$$(a)\ e^{t^2}, \qquad (b)\ f(t) = \begin{cases} 0, & t \text{ is even}, \\ 1, & \text{otherwise}. \end{cases} \qquad (c)\ e^{2t}.$$

Hint: The set of even numbers is a *negligible set* in the sense of integral.

Solution. First recall that *appropriate* means piecewise continuous on $[0, \infty)$ and of the exponential order. In particular, there exist positive constants M, α such that

$$|f(t)| \leq Me^{\alpha t}, \qquad t \geq 0.$$

In the list,

(a) e^{t^2} is not of the exponential order because for any $M, \alpha > 0$, $Me^{\alpha t} = e^{(\ln M)\alpha t}$ and $t^2 > (\ln M)\alpha t$ as long as $t > (\ln M)\alpha$. Hence, e^{t^2} is not appropriate.

(b) $f(t) = \begin{cases} 0, & t \text{ is even}, \\ 1, & \text{otherwise}. \end{cases}$ is appropriate since it is piecewise continuous and bounded, hence of the exponential order. Now, since the set of even numbers is negligible, the Laplace transform of $f(t)$ must be the same as that of the function $g(t) = 1$. Therefore,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} = \frac{1}{s}, \qquad s > 0.$$

(c) e^{2t} is appropriate.

$$\mathcal{L}\{e^{2t}\} = \frac{1}{s-2}, \qquad s > 2.$$

2. (20 points) (1) Use the derivative property of the Laplace transform to find $\mathcal{L}^{-1}\{s^{-4}\}$.

Solution. Note that

$$\mathcal{L}\{t\} = \frac{1}{s^2}.$$

Apply the derivative property successively twice, we obtain

$$\mathcal{L}\{t^2\} = -\frac{d}{ds}\frac{1}{s^2} = \frac{2}{s^3},$$
$$\mathcal{L}\{t^3\} = -\frac{d}{ds}\frac{2}{s^3} = \frac{6}{s^4}.$$

Therefore,

$$\mathcal{L}^{-1}\{s^{-4}\} = \frac{1}{6}t^3.$$

(2) Use the Laplace transform to find the function $f(t)$ which satisfies the integral equation

$$f(t) = t + \int_0^t f(x)\sin(t-x)dx.$$

Solution. Observe that the integral equation involves a convolution integral and thus can be rewritten as

$$f(t) = t + \sin t * f(t).$$

Now, apply the Laplace transform to both sides of this equality, then use the convolution formula, we have

$$F(s) = \frac{1}{s^2} + \frac{1}{1+s^2}F(s).$$

This is just

$$F(s) = \frac{1+s^2}{s^4} = \frac{1}{s^4} + \frac{1}{s^2}.$$

The inverse Laplace transform of $F(s)$ is now evident:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{6}t^3 + t.$$

3. (25 points) Consider the initial value problem describing the motion of a harmonic oscillator (without friction):

$$y'' + ky = g(t), \qquad y(0) = 1, y'(0) = 0.$$

where $k > 0$ is a constant and $g(t)$ is the forcing function.
(1) Use the Laplace transform to find the solution of the initial value problem. Your result could involve $g(t)$.

Solution. Apply the Laplace transform to both sides of the equation, we have

$$s^2Y(s) - s + kY(s) = G(s).$$

Thus,

$$Y(s) = \frac{G(s)}{s^2+k} + \frac{s}{s^2+k}.$$

By the inverse Laplace transforms, it follows that

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{\sqrt{k}} g(t) * \sin \sqrt{k}t + \cos \sqrt{k}t.$$

(2) Now, for the forcing function

$$g(t) = \sin ct, \qquad t \geq 0,$$

where $c > 0$ is a constant, use your result in part (1) to find the solution of the initial value problem above. For different values of c, does the solution converge/diverge/oscillate as $t \to \infty$?

Solution. By part (1), we have

$$\begin{aligned} y(t) &= \frac{1}{\sqrt{k}} \int_0^t \sin c(t-\tau) \sin \sqrt{k}\tau d\tau + \cos \sqrt{k}t \\ &= \frac{1}{\sqrt{k}} \int_0^t \frac{1}{2}(\cos(ct - (c+\sqrt{k})\tau) - \cos(ct - (c-\sqrt{k})\tau))d\tau + \cos \sqrt{k}t \end{aligned}$$

Case 1: $c = \sqrt{k}$

$$\begin{aligned} y(t) &= \frac{1}{2\sqrt{k}} \int_0^t (\cos(\sqrt{k}t - 2\sqrt{k}\tau) - \cos \sqrt{k}t)d\tau + \cos \sqrt{k}t \\ &= -\frac{1}{4k} \sin(\sqrt{k}t - 2\sqrt{k}\tau)\Big|_{\tau=0}^{t} - \frac{1}{2\sqrt{k}} t \cos \sqrt{k}t + \cos \sqrt{k}t \\ &= \frac{1}{2k} \sin \sqrt{k}t - \frac{1}{2\sqrt{k}} t \cos \sqrt{k}t + \cos \sqrt{k}t. \end{aligned}$$

Case 2: $c \neq \sqrt{k}$

$$\begin{aligned} y(t) &= -\frac{1}{2\sqrt{k}(c+\sqrt{k})} \sin(ct - (c+\sqrt{k})\tau)\Big|_{\tau=0}^{t} \\ &\quad + \frac{1}{2\sqrt{k}(c-\sqrt{k})} \sin(ct - (c-\sqrt{k})\tau)\Big|_{\tau=0}^{t} + \cos \sqrt{k}t \\ &= \frac{c}{\sqrt{k}} \frac{\sin \sqrt{k}t}{c^2-k} - \frac{\sin ct}{c^2-k} + \cos \sqrt{k}t. \end{aligned}$$

Solution in case 1 diverges and oscillates, solution in case 2 oscillates.

4. (20 points) Use the series method to find a *fundamental set of solutions* for the differential equation

$$(x^2+8)y'' + xy' + 2y = 0,$$

about $x = 0$. You only need to calculate terms up to the *third* power of x. What is a radius of convergence that you could claim about the series solutions, why?

Solution. First, we can rewrite the equation in the form

$$y'' + \frac{x}{x^2+8}y' + \frac{2}{x^2+8} = 0.$$

Since x^2+8 has the complex roots $r = \pm 2\sqrt{2}i$, both at the distance $2\sqrt{2}$ to the origin, by a theorem in class, we could claim that $x(x^2+8)^{-1}$ and $2(x^2+8)^{-1}$ are analytic on the interval $(-2\sqrt{2}, 2\sqrt{2})$, and the series solutions of the equation have radius of convergence being at least $2\sqrt{2}$.

Now we use the series method to solve the equation. Letting

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ...,$$

we also have

$$y'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + ...$$
$$y''(x) = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3...$$

Plugging this in the original equation, we have

$$(x^2+8)(2a_2+6a_3x+...) + x(a_1+2a_2x+3a_3x^2+...) + 2(a_0+a_1x+a_2x^2+a_3x^3+...)$$
$$= (16a_2+2a_0) + (48a_3+3a_1)x + ...$$
$$= 0.$$

Hence,

$$a_2 = -\frac{1}{8}a_0, \qquad a_3 = -\frac{1}{16}a_1.$$

Therefore, the series solution is

$$y(x) = a_0\Big(1 - \frac{1}{8}x^2 + ...\Big) + a_1\Big(x - \frac{1}{16}x^3 + ...\Big).$$

5. (10 points) Find the *fundamental set of solutions* of the differential equation

$$x^2y'' + xy' - 4y = 0.$$

Which one of the solutions could not be obtained by the Taylor series method at $x = 0$? Briefly explain why.

Solution. Realize that this is an Euler equation with $\alpha = 1, \beta = 4$. The polynomial

$$r^2 + (\alpha - 1)r + \beta = r^2 - 4$$

has roots

$$r = \pm 2.$$

Therefore,

$$|x|^r = |x|^{\pm 2},$$

and the two fundamental solutions are

$$x^2, \qquad x^{-2}.$$

In particular, x^{-2} is impossible to be obtained by the Taylor series method at $x = 0$, because its Taylor expansion at $x = 0$ does not exist.

6. (10 points) Determine whether the following two-point boundary value problem has solutions.

$$\begin{cases} y'' + 4y = \cos x, \\ y(0) = y(\frac{1}{4}\pi) = 0. \end{cases}$$

Solution. First find the general solutions of the given equation

$$y'' + 4y = \cos x.$$

It is easy to see that the homogeneous solutions are

$$y_h(x) = c_1 \cos 2x + c_2 \sin 2x.$$

Let a particular solution of the above equation be

$$y_p(x) = a\cos x + b\sin x.$$

We have

$$(-a + 4a)\cos x + (-b + 4a)\sin x = \cos x.$$

Hence,

$$a = \frac{1}{3}, \qquad b = 0.$$

Solutions are

$$y(x) = y_h(x) + y_p(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3}\cos x.$$

Now take into account the boundary values:

$$y(0) = c_1 + \frac{1}{3} = 0,$$
$$y\left(\frac{1}{4}\pi\right) = c_2 + \frac{\sqrt{2}}{6}.$$

Therefore,

$$c_1 = -\frac{1}{3}, \qquad c_2 = -\frac{\sqrt{2}}{6}.$$

And the boundary value problem has a unique solution.