PRACTICE EXAM II, MATH 353

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. You have **75 minutes** to answer all the questions. *Good Luck* !

Laplace Transforms:

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \mathcal{L}\{\sin at\} = \frac{a}{a^2+s^2}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{\cos at\} = \frac{s}{a^2+s^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \qquad \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s) \qquad \mathcal{L}\{f'(t)\} = -f(0)+sF(s)$$

$$\mathcal{L}\{\delta(t)\} = 1 \qquad \mathcal{L}\{(f*g)(t)\} = F(s)G(s)$$

Question	Max. Points	Score
1	15	
2	20	
3	25	
4	20	
5	10	
6	10	
Total	100	

1. (15 points) Which of the following functions are *appropriate* for the Laplace transform. For those that are appropriate, find the Laplace transform.

$$(a)\ e^{t^2}, \qquad (b)\ f(t) = \begin{cases} 0, & t \text{ is even}, \\ 1, & \text{otherwise}. \end{cases} \qquad (c)\ e^{2t}.$$

Hint: The set of even numbers is a *negligible set* in the sense of integral.

2. (20 points) (1) Use the derivative property of the Laplace transform to find $\mathcal{L}^{-1}\{s^{-4}\}$.

(2) Use the Laplace transform to find the function $f(t)$ which satisfies the integral equation

$$f(t) = t + \int_0^t f(x)\sin(t-x)dx.$$

3. (25 points) Consider the initial value problem describing the motion of a harmonic oscillator (without friction):

$$y'' + ky = g(t), \qquad y(0) = 1, y'(0) = 0.$$

where $k > 0$ is a constant and $g(t)$ is the forcing function.

(1) Use the Laplace transform to find the solution of the initial value problem. Your result could involve $g(t)$.

(2) Now, for the forcing function

$$g(t) = \sin ct, \qquad t \geq 0,$$

where $c > 0$ is a constant, use your result in part (1) to find the solution of the initial value problem above. For different values of c, does the solution converge/diverge/oscillate as $t \to \infty$?

4. (20 points) Use the series method to find a *fundamental set of solutions* for the differential equation

$$(x^2 + 8)y'' + xy' + 2y = 0,$$

about $x = 0$. You only need to calculate terms up to the *third* power of x. What is a radius of convergence that you could claim about the series solutions, why?

5. (10 points) Find the *fundamental set of solutions* of the differential equation

$$x^2y'' + xy' - 4y = 0.$$

Which one of the solutions could not be obtained by the Taylor series method at $x = 0$? Briefly explain why.

6. (10 points) Determine whether the following two-point boundary value problem has solutions.

$$\begin{cases} y'' + 4y = \cos x, \\ y(0) = y(\frac{1}{4}\pi) = 0. \end{cases}$$