

PRACTICE EXAM I SOLUTIONS

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name:

Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. You have **75 minutes** to answer all the questions. *Good Luck!*

Useful Formulas:

$$y_p(x) = -y_1(x) \int_{x_0}^x \frac{y_2(s)g(s)}{W(y_1, y_2)} ds + y_2(x) \int_{x_0}^x \frac{y_1(s)g(s)}{W(y_1, y_2)} ds,$$

or

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

You are responsible for identifying correctly the situation in which these formulas can be applied.

Question	Max. Points	Score
1	15	
2	20	
3	15	
4	20	
5	15	
6	15	
Total	100	

1. (15 points) Find the general solution of the differential equation

$$2xy^3 + 3(x^2 + 1)y^2 \frac{dy}{dx} = \cos x.$$

Solution. First write the equation in the form $M(x, y) + N(x, y) \frac{dy}{dx} = 0$:

$$(2xy^3 - \cos x) + 3(x^2 + 1)y^2 \frac{dy}{dx} = 0.$$

Now,

$$\frac{\partial M}{\partial y} = 6xy^2 = \frac{\partial N}{\partial x}.$$

Thus the equation is exact, and we have

$$F(x, y) = \int (2xy^3 - \cos x) dx + h(y) = x^2y^3 - \sin x + h(y).$$

Furthermore,

$$3(x^2 + 1)y^2 = N(x, y) = \frac{\partial F}{\partial y} = 3x^2y^2 + h'(y).$$

Hence,

$$h'(y) = 3y^2$$

and a choice of $h(y)$ is

$$h(y) = y^3.$$

Therefore, the general solution of the original equation is

$$F(x, y) = x^2y^3 - \sin x + y^3 = C,$$

for arbitrary constants C .

2. (20 points) Solve the initial value problem

$$\begin{cases} y' &= y^2(1+x) \\ y(0) &= y_0 \end{cases}$$

and find the domain of definition of the solution. Your result should cover all possible values of y_0 .

Solution. First note that when $y \neq 0$, the equation is separable:

$$-(1+x) + \frac{1}{y^2}y' = 0.$$

This is equivalent to

$$\frac{d}{dx} \left(-x - \frac{1}{2}x^2 - \frac{1}{y} \right) = 0.$$

Therefore, we have the general solution

$$x + \frac{1}{2}x^2 + \frac{1}{y} = C,$$

or

$$y = \left(C - x - \frac{1}{2}x^2 \right)^{-1}.$$

Of course, in this case, y is never vanishing. Hence, for $y_0 \neq 0$, $C = y_0^{-1}$.

The other case is when $y_0 = 0$, and it is evident that $y(x) = 0$ is a solution of the original equation with this initial value.

Therefore,

$$y(x) = \begin{cases} \left(y_0^{-1} - x - \frac{1}{2}x^2 \right)^{-1}, & y_0 \neq 0, \\ 0, & y_0 = 0. \end{cases}$$

To determine the domain of definition of these solutions, one should note that the initial value is set at $x_0 = 0$. For $y_0 \neq 0$, one should only consider the interval on which $y_0^{-1} - x - \frac{1}{2}x^2$ is non-zero, i.e., $\frac{2}{y_0} + 1 \neq (x+1)^2$. The cases are the following:

Case 1: $y_0 = 0$. Here, $y(x) = 0$ and the domain of definition is $(-\infty, \infty)$.

Case 2: $y_0 > 0$. $y_0^{-1} - x - \frac{1}{2}x^2$ is zero when $x = -1 \pm (2/y_0 + 1)^{\frac{1}{2}}$ (one positive, the other negative). Hence, the domain of definition is $(-1 - (2/y_0 + 1)^{\frac{1}{2}}, -1 + (2/y_0 + 1)^{\frac{1}{2}})$.

Case 3: $-2 < y_0 < 0$. $y_0^{-1} - x - \frac{1}{2}x^2$ is always nonzero. Hence the domain of definition is $(-\infty, \infty)$.

Case 4: $y_0 \leq -2$. $y_0^{-1} - x - \frac{1}{2}x^2$ is zero when $x = -1 \pm (2/y_0 + 1)^{\frac{1}{2}}$ (both negative). Hence, the domain of definition is $(-1 + (2/y_0 + 1)^{\frac{1}{2}}, \infty)$.

3. (15 points) Use a method you learned in homework to find the general (implicit) solutions of the equation

$$y' = \frac{-2x + 5y}{x + 2y}.$$

Solution. Observe that both the denominator and numerator on the right hand side of the equation are homogeneous functions of the same degree. Consider the substitution

$$z = \frac{y}{x}.$$

Thus,

$$y' = (zx)' = z'x + z.$$

The equation becomes

$$z'x + z = \frac{-2 + 5y/x}{1 + 2y/x} = \frac{-2 + 5z}{1 + 2z},$$

which simplifies to

$$z'x = \frac{-2 + 4z - 2z^2}{1 + 2z}.$$

This equation in z is clearly separable when $z \neq 1$:

$$\frac{1 + 2z}{-2(z - 1)^2} z' = \frac{1}{x}.$$

Note that the left hand side of the equality above is

$$\frac{1 + 2z}{-2(z - 1)^2} z' = \frac{2(z - 1) + 3}{-2(z - 1)^2} z' = \left(-\frac{1}{z - 1} - \frac{3}{2(z - 1)^2} \right) z' = \left(-\ln|z - 1| + \frac{3}{2} \frac{1}{z - 1} \right)'$$

Therefore,

$$-\ln|z - 1| + \frac{3}{2} \frac{1}{z - 1} = \ln|x| + C,$$

i.e., plugging in $z = y/x$,

$$-\ln|y - x| + \frac{3}{2} \frac{x}{y - x} = C,$$

for arbitrary constant C .

When $z = 1$, we have that $y = x$ is another solution of the original equation.

4. (20 points) Given that $y_1(x) = x$ and $y_2(x) = x^{-1}$ are to solutions of the differential equation

$$x^2y'' + xy' - y = 0,$$

find the general solution of

$$x^2y'' + xy' - y = x, \quad x > 0.$$

Solution. Use the method of variation of parameters. Note that using our notations, $g(x) = \frac{1}{x}$ in this case.

First, we calculate the Wronskian

$$W(y_1, y_2) = \det \begin{pmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{pmatrix} = -2x^{-1}.$$

Using the formula given in this exam, we have

$$\begin{aligned} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} &= \frac{1}{W(x, y)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ g(x) \end{pmatrix} \\ &= -\frac{1}{2}x \begin{pmatrix} -x^{-2} & -x^{-1} \\ -1 & x \end{pmatrix} \begin{pmatrix} 0 \\ x^{-1} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} x^{-1} \\ -x \end{pmatrix}. \end{aligned}$$

Therefore, a choice of u_1, u_2 is

$$u_1(x) = \frac{1}{2} \ln x, \quad u_2(x) = -\frac{1}{4}x^2.$$

Thus, a particular solution of the original equation is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) = \frac{1}{2}x \ln x - \frac{1}{4}x.$$

The general solution is

$$y(x) = c_1x + c_2x^{-1} + \frac{1}{2}x \ln x,$$

for arbitrary constants c_1, c_2 .

5. (15 points) In class, we learned that for *homogeneous* linear second order ODEs, one could start with one non-zero solution, apply the method of *reduction of order*, and obtain a fundamental set of solutions for the equation. In this problem, you will see how reduction of order could also apply to finding a particular solution of a non-homogeneous equation.

(1) Consider the equation in problem 4,

$$x^2y'' + xy' - y = x, \quad x > 0.$$

We have seen that $y_h(x) = x$ is a solution of the corresponding homogeneous equation. Now, assume that $v(x)y_h(x)$ is a solution of the above non-homogeneous equation for some function $v(x)$. Show that $v(x)$ must satisfy the linear equation

$$x^2v'' + 3xv' = 1.$$

Solution. Suppose that $vy_h = vx$ is a solution of the non-homogeneous equation, we have

$$\begin{aligned} x &= x^2(vx)'' + x(vx)' - (vx) \\ &= x^2(v''x + 2v') + x(v'x + v) - vx \\ &= x^3v'' + 3x^2v'. \end{aligned}$$

This is exactly

$$x^2v'' + 3xv' = 1,$$

for $x > 0$.

(2) Find such a function v by reducing the order of the equation $x^2v'' + 3xv' = 1$.

Solution. Reduce the order of the equation by taking $z = v'$. Now the equation becomes

$$x^2z' + 3xz = 1.$$

Note that this equation is linear, thus we could use the method of integrating factors to solve it. Let $w(x) = \mu(x)z(x)$, then $w(x)$ satisfies the equation

$$\begin{aligned} w' &= \mu'z + \mu z' = \frac{\mu'}{\mu}w + \mu(x^{-2} - 3x^{-1}z) \\ &= \frac{\mu'}{\mu}w - 3x^{-1}w + x^{-2}\mu. \end{aligned}$$

We want to choose $\mu(x)$ so that

$$\frac{\mu'}{\mu} = \frac{3}{x},$$

i.e.,

$$\ln |\mu| = 3 \ln |x|.$$

Evidently, we could choose

$$\mu(x) = x^3.$$

Now,

$$w' = x^{-2}\mu = x,$$

and

$$w = \frac{1}{2}x^2.$$

Finally,

$$z = \frac{w}{\mu} = \frac{1}{2x}.$$

By $v' = z$, a choice of $v(x)$ is thus

$$v = \frac{1}{2} \ln x.$$

(3) Find a particular solution of the original equation. Does it agree with what you obtained in problem 4?

Solution. By part (2), we conclude that a particular solution of the equation $x^2y'' + xy' - y = x$ is

$$v(x)y_h(x) = \frac{1}{2}x \ln x.$$

This agrees with the result we obtained in problem 4.

6. (15 points) Suppose that, under natural conditions, the population of fish in a pond obeys the model

$$\dot{p} = p(4 - p).$$

Now, suppose that people start harvesting from the pond at a continuous rate of c . (Assume that the units are consistent.)

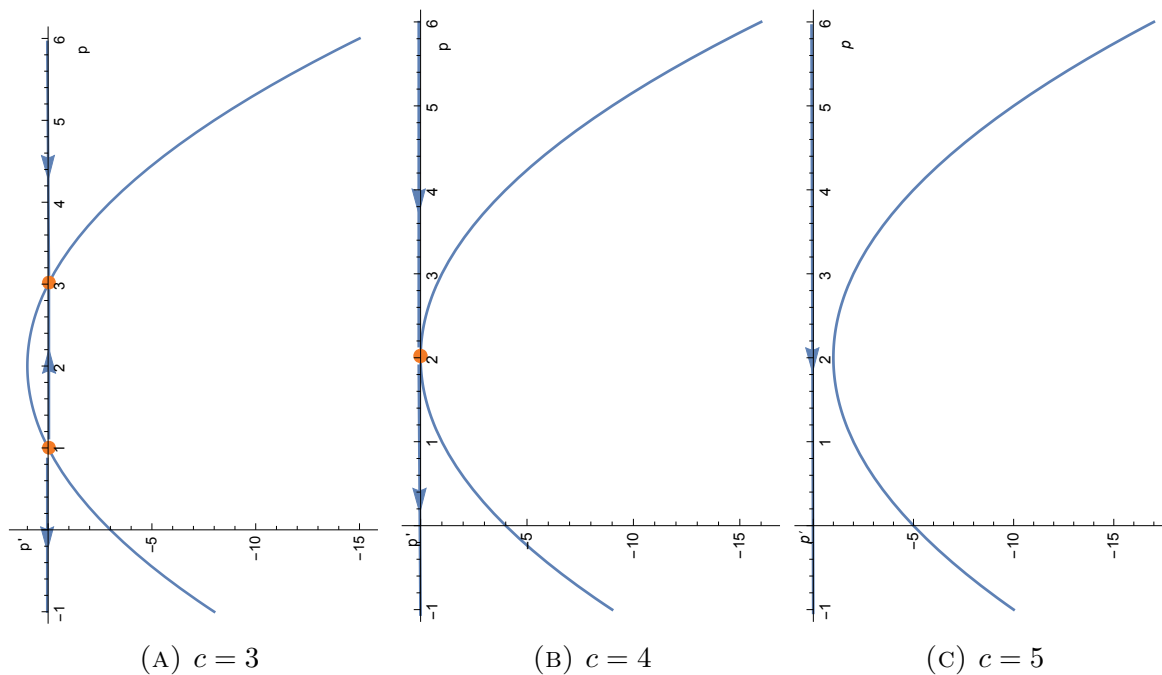
(1) Establish a model (differential equation) which describes the population of fish in the pond after the harvesting has started.

Solution. According to the description, we have

$$\dot{p} = p(4 - p) - c.$$

(2) For $c = 3, 4, 5$ each, use the phase line analysis to find the equilibria (if any) of your equation. Are they stable or unstable?

Solution.



Conclusion:

$c = 3$: Two equilibria, $p = 1$ unstable and $p = 3$ stable.

$c = 4$: Only one equilibrium, $p = 2$, unstable.

$c = 5$: No equilibria.