PRACTICE EXAM I, MATH 353

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: You may not use any notes, books, calculators or computers. Moreover, you must also show the work you did to arrive at the answer to receive full credit. If you are using a theorem to draw some conclusions, quote the result. You have **75 minutes** to answer all the questions. *Good Luck* !

Useful Formulas:

$$y_p(x) = -y_1(x)\int_{x_0}^{x} \frac{y_2(s)g(s)}{W(y_1,y_2)}ds + y_2(x)\int_{x_0}^{x} \frac{y_1(s)g(s)}{W(y_1,y_2)}ds,$$

or

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.$$

You are responsible for identifying correctly the situation in which these formulas can be applied.

Question	Max. Points	Score
1	15	
2	20	
3	15	
4	20	
5	15	
6	15	
Total	100	

1. (15 points) Find the general solutions of the differential equation

$$2xy^3 + 3(x^2+1)y^2\frac{dy}{dx} = \cos x.$$

2. (20 points) Solve the initial value problem

$$\begin{cases} y' & = y^2(1+x) \\ y(0) & = y_0 \end{cases}$$

and find the domain of definition of the solution. Your result should cover all possible values of y_0.

3. (15 points) Use a method you learned in homework to find the general (implicit) solutions of the equation

$$y' = \frac{-2x + 5y}{x + 2y}.$$

4. (20 points) Given that $y_1(x) = x$ and $y_2(x) = x^{-1}$ are to solutions of the differential equation

$$x^2y'' + xy' - y = 0,$$

find the general solution of

$$x^2y'' + xy' - y = x, \qquad x > 0.$$

5. (15 points) In class, we learned that for *homogeneous* linear second order ODEs, one could start with one non-zero solution, apply the method of *reduction of order*, and obtain a fundamental set of solutions for the equation. In this problem, you will see how reduction of order could also apply to finding a particular solution of a non-homogeneous equation.

(1) Consider the equation in problem 4,

$$x^2y'' + xy' - y = x, \qquad x > 0.$$

We have seen that $y_h(x) = x$ is a solution of the corresponding homogeneous equation. Now, assume that $v(x)y_h(x)$ is a solution of the above non-homogeneous equation for some function $v(x)$. Show that $v(x)$ must satisfy the linear equation

$$x^2v'' + 3xv' = 1.$$

(2) Find such a function v by reducing the order of the equation $x^2v'' + 3xv' = 1$.

(3) Find a particular solution of the original equation. Does it agree with what you obtained in problem 4?

6. (15 points) Suppose that, under natural conditions, the population of fish in a pond obeys the model

$$\dot{p} = p(4 - p).$$

Now, suppose that people start harvesting from the pond at a continuous rate of c. (Assume that the units are consistent.)

(1) Establish a model (differential equation) which describes the population of fish in the pond after the harvesting has started.

(2) For $c = 3, 4, 5$ each, use the phase line analysis to find the equilibria (if any) of your equation. Are they stable or unstable?