FINAL EXAM, MATH 353 SUMMER I 2015

9:00am-12:00pm, Thursday, June 25

I have neither given nor received any unauthorized help on this exam and I have conducted myself within the guidelines of the Duke Community Standard.

Name: Signature:

Instructions: This exam contains **12 pages** and **10 problems** with a table of Laplace transform at the end. You have **180 minutes** to answer all the questions. You may use a calculator or a review sheet, front and back, written in your own handwriting. Throughout the exam, show your work with clear reasoning and calculation. If you are using a theorem to draw some conclusions, quote the result. If you do not completely solve a problem, explain what you understand about it. No collaboration on this exam is allowed. *Good luck* !

Problems	Points	Grade
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total	200	

1. (20 points) Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ of the following functions:

(1) $F(s) = \dfrac{s-2}{s^2+2s+10}$

Solution.
$$\frac{s-2}{s^2+2s+10} = \frac{s-2}{(s+1)^2+9} = \frac{s+1}{(s+1)^2+9} - \frac{3}{(s+1)^2+9}.$$
Thus, by the first shift theorem,
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t}(\cos 3t - \sin 3t).$$

(2) $F(s) = \dfrac{e^{-s}}{s^3}$

Solution. Note that $\mathcal{L}\{t^2\} = \frac{2!}{s^3}$, thus
$$\mathcal{L}^{-1}\{s^{-3}\} = \frac{1}{2}t^2.$$
By the second shift theorem, we have
$$f(t) = \frac{1}{2}u_1(t)(t-1)^2.$$

(3) $F(s) = \dfrac{1}{(s^2-4)(s^2-9)}$

Solution. Note that (by partial fractions)
$$\frac{1}{(s^2-4)(s^2-9)} = \frac{1}{5}\Big(\frac{1}{s^2-9} - \frac{1}{s^2-4}\Big).$$
Therefore,
$$f(t) = -\frac{1}{10}\sinh 2t + \frac{1}{15}\sinh 3t.$$

2. (20 points) Find the solution of the initial value problem

$$y'' + 4y = 8\sinh(2t) + \delta(t - 2\pi) \qquad y(0) = y'(0) = 0,$$

where $\delta(t)$ is the Dirac delta function and $\sinh(t)$ is the hyperbolic sine function.

Solution. Take the Laplace transform on both sides of the equation and note that the initial values are zero. We obtain

$$(s^2 + 4)Y(s) = \frac{16}{s^2 - 4} + e^{-2\pi s}.$$

Thus,

$$\begin{aligned} Y(s) &= \frac{16}{(s^2 + 4)(s^2 - 4)} + e^{-2\pi s}\frac{1}{s^2 + 4} \\ &= \frac{2}{s^2 - 4} - \frac{2}{s^2 + 4} + \frac{1}{2}e^{-2\pi s}\frac{2}{s^2 + 4}. \end{aligned}$$

Now, taking the inverse Laplace transform, we have

$$\begin{aligned} y(t) &= \sinh 2t - \sin 2t + \frac{1}{2}u_{2\pi}(t)\sin 2(t - 2\pi) \\ &= \sinh 2t - \sin 2t + \frac{1}{2}u_{2\pi}(t)\sin 2t. \end{aligned}$$

3. (20 points) Consider the first order ordinary differential equation

$$e^x(x+1) + (ye^y - xe^x)\frac{dy}{dx} = 0.$$

(1) Find an integrating factor $g(y)$ which makes this equation *exact*.

Solution. The integrating factor $g(y)$ should satisfy

$$\frac{\partial[g(y)e^x(x+1)]}{\partial y} = \frac{\partial[g(y)(ye^y - xe^x)]}{\partial x},$$

which is just

$$g'(y)(e^x(x+1)) = g(y)(-e^x - xe^x).$$

Hence,

$$\frac{g'}{g} = -1,$$

and we can choose

$$g(y) = e^{-y}.$$

(2) Find the (implicit) solution of the differential equation above with $y(0) = 1$.

Solution. Multiplying the equation by e^{-y}, we have

$$e^{x-y}(x+1) + (y - xe^{x-y})\frac{dy}{dx} = 0.$$

Thus,

$$\begin{aligned} F(x,y) &= \int e^{x-y}(x+1)dx + h(y) \\ &= e^{-y}xe^x + h(y). \end{aligned}$$

Furthermore, we need

$$y - xe^{x-y} = \frac{\partial F}{\partial y}(x,y) = -e^{-y}xe^x + h'(y).$$

It follows that

$$h'(y) = y,$$

and we can choose $h(y) = \frac{1}{2}y^2$.
Therefore, the general solution is

$$xe^{x-y} + \frac{1}{2}y^2 = C,$$

for some constant C. Using the given initial value, we have $C = \frac{1}{2}$.
We conclude that the solution of the initial value problem is

$$xe^{x-y} + \frac{1}{2}y^2 = \frac{1}{2}.$$

4. (20 points) Find the general solution of the equation

$$(x+1)y'' - xy' - y = 0, \qquad x > -1$$

given that $y(x) = e^x$ is a solution. Your final answer could involve an indefinite integral.

Solution. Use the method of Reduction of order: consider a solution of the form $v(x)y(x) = v(x)e^x$ and find $v(x)$. We have

$$(x+1)(v''e^x + 2v'e^x + ve^x) - x(v'e^x + ve^x) - ve^x = 0,$$

which is equivalent to

$$(x+1)v'' + (x+2)v' = 0.$$

Letting $z = v'$, we have

$$\frac{z'}{z} = -\frac{x+2}{x+1} = -1 - \frac{1}{x+1}.$$

Hence,

$$\ln|z| = -x - \ln(x+1).$$

We can choose

$$z(x) = \frac{e^{-x}}{x+1},$$

and

$$v(x) = \int \frac{e^{-x}}{x+1} dx.$$

Therefore, the general solution of the equation is

$$y(x) = c_1 e^x + c_2 v(x) e^x = e^x \Big(c_1 + c_2 \int \frac{e^{-x}}{x+1} dx \Big).$$

5. (20 points) Use the series method to solve the initial value problem

$$y'' + 3xy' + e^x y = 2x, \qquad y(0) = 1,\ y'(0) = -1.$$

You need only to calculate terms up to the *fourth* power of x.
Note: The Taylor expansion of e^x at $x = 0$ is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...$$

Solution. Let

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ...$$

We have

$$y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + ...$$
$$y''(x) = 2a_2 + 6a_3 x + 12a_4 x^2 + ...$$

Therefore, expanding all the functions in the equation in their Taylor series about $x = 0$, we obtain

$$(2a_2 + 6a_3 x + 12a_4 x^2 + ...) + 3x(a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + ...) +$$
$$(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ...)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + ...) = 2x.$$

If we compare the coefficients of the terms $1, x, x^2$ on both sides of the equation, we would obtain

$$2a_2 + a_0 = 0,$$
$$6a_3 + 3a_1 + a_0 + a_1 = 2,$$
$$12a_4 + 6a_2 + a_2 + a_1 + \frac{1}{2}a_0 = 0.$$

Note that the initial values are just

$$a_0 = 1, \qquad a_1 = -1.$$

Therefore, from the above equalities,

$$a_2 = -\frac{1}{2},$$
$$a_3 = \frac{2 - 4a_1 - a_0}{6} = \frac{5}{6},$$
$$a_4 = \frac{-\frac{1}{2}a_0 - a_1 - 7a_2}{12} = \frac{1}{3}.$$

Therefore, the series solution is

$$y(x) = 1 - x - \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{1}{3}x^4 + ...$$

6. (20 points) Show by calculation that the Fourier series for the function

$$f(x) = e^x, \qquad -\pi \le x < \pi, \; f(x) = f(x+2\pi)$$

is

$$g(x) = \frac{\sinh \pi}{\pi}\Big(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos nx - n \sin nx)\Big).$$

What is the value of $g(\pi)$? Now find the limit of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$

Solution. Note that the period is 2π. If we let $g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_2 \sin nx)$, then

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} = \frac{2}{\pi}\sinh \pi, \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx dx, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx dx, \end{aligned}$$

Using integration by parts, we could evaluate

$$\int e^x \cos nx dx = e^x \cos nx + n \int e^x \sin nx dx, \qquad \int e^x \sin nx dx = e^x \sin nx - n \int e^x \sin nx dx.$$

If we let $A = \int_{-\pi}^{\pi} e^x \cos nx dx$, $B = \int_{-\pi}^{\pi} e^x \sin nx dx$, then the above equality tells us

$$\begin{cases} A = (e^{\pi} - e^{-\pi}) \cos n\pi + nB, \\ B = -nA. \end{cases}$$

Therefore,

$$\begin{aligned} A &= \frac{2}{1+n^2}\cos n\pi \sinh \pi = \frac{2}{1+n^2}(-1)^n \sinh \pi, \\ B &= -nA = (-n)\frac{2}{1+n^2}(-1)^n \sinh \pi. \end{aligned}$$

Plugging this in the expression of $g(x)$ gives the Fourier expansion.

By the Fourier convergence theorem and the periodicity of $f(x)$, we know that

$$g(\pi) = \frac{1}{2}(e^{\pi^+} + e^{\pi^-}) = \frac{1}{2}(e^{-\pi} + e^{\pi}) = \cosh \pi.$$

On the other hand,

$$\begin{aligned} g(\pi) &= \frac{\sinh \pi}{\pi}\Big(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}(\cos n\pi - n \sin n\pi)\Big) \\ &= \frac{\sinh \pi}{\pi}\Big(1 + 2\sum_{n=1}^{\infty} \frac{1}{1+n^2}\Big). \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}(\pi \coth \pi - 1).$$

7. (20 points) Consider the initial-boundary value problem:

$$\begin{cases} u_{xx} = u_t - \sin x, & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = 3\sin 2x + 6\sin 5x, & 0 \le x \le \pi \end{cases}$$

(1) Find the **steady state** solution of this equation, i.e., a solution $w(x,t) = w(x)$ (does not depend on t) which satisfies the equation and the boundary values.

Solution. By the definition of *steady state*, we have that $w(x)$ satisfies

$$\begin{cases} w''(x) = -\sin x, & 0 < x < \pi, \\ w(0) = w(\pi) = 0. \end{cases}$$

It is easy to see (or by integrating twice) that

$$w(x) = \sin x, \qquad 0 < x < \pi.$$

(2) Find the solution $u(x,t)$ of the initial-boundary value problem.
Hint: Which initial-boundary value problem does $v(x,t) = u(x,t) - w(x)$ satisfy?

Solution. By the linearity of the equation, $v(x,t) = u(x,t) - w(x)$ satisfies the initial-boundary value problem:

$$\begin{cases} v_{xx} = v_t, & 0 < x < \pi,\ t > 0 \\ v(0,t) = v(\pi,t) = 0, & t > 0 \\ v(x,0) = -\sin x + 3\sin 2x + 6\sin 5x, & 0 \le x \le \pi \end{cases}$$

Note that this is just the heat equation with the temperature at both ends set to be zero. We know that the solution are in the form ($\alpha = 1, L = \pi$)

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin nx\ e^{-n^2 t}.$$

The initial value for $v(x,t)$ is

$$v(x,0) = \sum_{n=1}^{\infty} b_n \sin nx = -\sin x + 3\sin 2x + 6\sin 5x.$$

Therefore

$$b_1 = -1, \qquad b_2 = 3, \qquad b_5 = 6, \qquad b_n = 0\ (n \neq 1, 2, 5),$$

and

$$v(x,t) = -e^{-t}\sin x + 3e^{-4t}\sin 2x + 6e^{-25t}\sin 5x.$$

We conclude that

$$u(x,t) = v(x,t) + w(x) = (1 - e^{-t})\sin x + 3e^{-4t}\sin 2x + 6e^{-25t}\sin 5x.$$

8. (20 points) Find the solution $u(x,t)$ of the following wave equation problem:

$$\begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = 2\sin 3x, \ u_t(x,0) = \sin 4x, & 0 \le x \le \pi \end{cases}$$

Solution. We consider instead the following two initial-boundary value problems

$$\text{(I)} \begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = 2\sin 3x, \ u_t(x,0) = 0, & 0 \le x \le \pi \end{cases}$$

and

$$\text{(II)} \begin{cases} u_{xx} = u_{tt}, & 0 < x < \pi, t > 0 \\ u(0,t) = u(\pi,t) = 0, & t > 0 \\ u(x,0) = 0, \ u_t(x,0) = \sin 4x, & 0 \le x \le \pi \end{cases}$$

From our discussion in class, solutions of (I) and (II) respectively take the form ($L = \pi, c = 1$)

$$u^{\mathrm{I}}(x,t) = \sum_{n=1}^{\infty} d_n \sin nx \cos nt,$$

and

$$u^{\mathrm{II}}(x,t) = \sum_{n=1}^{\infty} k_n \sin nx \sin nt.$$

Now the non homogeneous parts of the initial conditions tell us

$$u^{\mathrm{I}}(x,0) = \sum_{n=1}^{\infty} d_n \sin nx = 2\sin 3x,$$

$$u_t^{\mathrm{II}}(x,0) = \sum_{n=1}^{\infty} n k_n \sin nx = \sin 4x.$$

Thus,

$$d_3 = 2, \qquad k_4 = \frac{1}{4},$$

and the rest of the coefficients are all zero.
We conclude that, by linearity,

$$u(x,t) = u^{\mathrm{I}}(x,t) + u^{\mathrm{II}}(x,t) = 2\sin 3x \cos 3t + \frac{1}{4}\sin 4x \sin 4t.$$

9. (20 points) Find the solution $u(r, \theta)$ of the following Laplace equation in the *semicircular* region $r < 1,\ 0 < \theta < \pi$:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, & 0 \le r < 1,\ 0 < \theta < \pi \\ u(r,0) = u(r,\pi) = 0, & 0 \le r < 1 \\ u(1,\theta) = f(\theta), & 0 \le \theta \le \pi, \end{cases}$$

assuming that $u(r, \theta)$ is single-valued and bounded in the given region.
Note: You are allowed to quote intermediate results in our solution of a similar problem with the region being *circular*.

Solution. Same as in the case when the domain is circular, the separation of variable will give us two ODEs:

$$\begin{cases} r^2R'' + rR' - \lambda R = 0, \\ \Theta'' + \lambda\Theta = 0. \end{cases}$$

Now the boundary conditions $u(r, 0) = u(r, \pi) = 0$ give us

$$R(r)\Theta(0) = R(r)\Theta(\pi) = 0,$$

which is just

$$\Theta(0) = \Theta(\pi) = 0.$$

Therefore, $\Theta(\theta)$ satisfies the familiar two-point boundary value problem

$$\begin{cases} \Theta'' + \lambda\Theta = 0, \\ \Theta(0) = \Theta(\pi) = 0. \end{cases}$$

We know that the eigenvalues and eigenfunctions of this problem are

$$\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2, \qquad \Theta_n(\theta) = \sin nx.\ (n = 1, 2, 3, ...)$$

Correspondingly, $R_n(r)$ satisfies the Euler equation

$$r^2R'' + rR' - n^2R = 0,$$

and its general solution takes the form

$$R_n(r) = c_1 r^n + c_2 r^{-n}.$$

As in the circular case, we must have $c_2 = 0$ in order for the solution to be bounded, particularly as $r \to 0^+$. Therefore,

$$u_n(r, \theta) = R_n(r)\Theta_n(\theta) = r^n \sin n\theta.$$

And by the principle of superposition,

$$u(r, \theta) = \sum_{n=1}^{\infty} k_n u_n(r, \theta) = \sum_{n=1}^{\infty} k_n r^n \sin n\theta.$$

Now, take into consideration of the non-homogeneous boundary value, we have

$$f(\theta) = u(1, \theta) = \sum_{n=1}^{\infty} k_n \sin n\theta, \qquad 0 \le \theta \le \pi.$$

Therefore, the (sine) Fourier coefficients k_n are

$$k_n = \frac{2}{\pi}\int_0^{\pi} f(\theta) \sin n\theta d\theta.$$

This completes the solution.

10. (20 points) Consider the boundary value problem

$$\begin{cases} [(1+x^2)y']' + y = \lambda(1+x^2)y \\ y(0) - y'(1) = 0, \quad y'(0) + 2y(1) = 0. \end{cases}$$

Let L be defined by $L[y] = [(1+x^2)y']' + y$. Prove that for any $u(x), v(x)$ satisfying the *boundary conditions* above,

$$\langle L[u], v\rangle_H = \langle u, L[v]\rangle_H,$$

where $\langle u, v\rangle_H$ denotes the *Hermitian* L^2*-inner product* of u, v on the interval $[0, 1]$.
Hint: You may use integration by parts *twice*.

Solution. By the definition of L and the Hermitian L^2-inner product, we have

$$\begin{aligned}
\langle L[u], v\rangle_H &= \int_0^1 [(1+x^2)u']' \, \overline{v} dx \\
&= \int_0^1 \overline{v} d[(1+x^2)u'] \\
&= \overline{v}(1+x^2)u'\Big|_0^1 - \int_0^1 \overline{v'}(1+x^2)u' dx \\
&= \overline{v}(1+x^2)u'\Big|_0^1 - \int_0^1 \overline{v'}(1+x^2) du(x) \\
&= \overline{v}(1+x^2)u'\Big|_0^1 - \overline{v'}(1+x^2)u\Big|_0^1 + \int_0^1 \overline{[(1+x^2)v']'} \, u dx \\
&= \overline{v}(1+x^2)u'\Big|_0^1 - \overline{v'}(1+x^2)u\Big|_0^1 + \langle u, L[v]\rangle_H.
\end{aligned}$$

Thus, to complete the proof, we need only to show

$$\overline{v}(1+x^2)u'\Big|_0^1 - \overline{v'}(1+x^2)u\Big|_0^1 = 0.$$

In fact, the boundary values (which are not separated) tell us

$$u'(1) = u(0), \; v'(1) = v(0), \qquad 2u(1) = -u'(0), \; 2v(1) = -v'(0).$$

Therefore,

$$\begin{aligned}
&\overline{v}(1+x^2)u'\Big|_0^1 - \overline{v'}(1+x^2)u\Big|_0^1 \\
=&2\overline{v(1)} \; u'(1) - \overline{v(0)} \; u'(0) - 2\overline{v'(1)} \; u(1) + \overline{v'(0)} \; u(0) \\
=&-\overline{v'(0)} \; u(0) - \overline{v(0)} \; u'(0) + u'(0) \; \overline{v(0)} + \overline{v'(0)} \; u(0) \\
=&0.
\end{aligned}$$

This completes the proof.

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
t^n $(n > 0$, integer)	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, \quad s > 0$
$\cosh at$	$\frac{s}{s^2-a^2}, \quad s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
$e^{ct}f(t)$	$F(s-c)$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - ... - f^{(n-1)}(0)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$